Fluctuations of Chaotic Random Variables

Theoretical Foundations
and Geometric Applications

Giovanni Peccati (Luxembourg University)

Santander: July 17-21, 2017

OVERVIEW, I

- ⋆ Survey of the combinatorial structure of recently developed **probabilistic limit theorems**, based on several improvements of the **combinatorial method of moments and cumulants**.
- ⋆ Applications of a geometric flavour: **random geometric graphs, random tessellations, excursions of random fields,** ...
- ⋆ Other techniques/tools involved: **Gaussian analysis, Malliavin calculus of variations, Markov semigroups, Stein's method, Chen-Stein method,** ...

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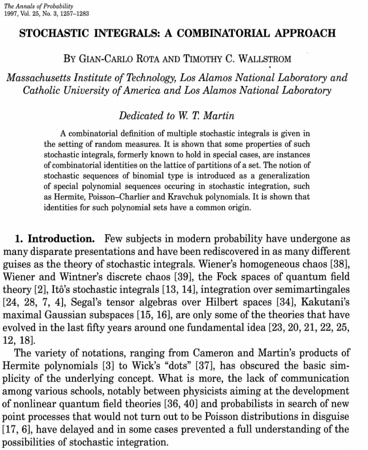


The Annals of Probability
1997, Vol. 25, No. 3, 1257–1283

STOCHASTIC INTEGRALS: A COMBINATORIAL APPROACH

BY GIAN-CARLO ROTA AND TIMOTHY C. WALLSTROM

Massachusetts Institute of Technology, Los Alamos National Laboratory and Catholic University of America and Los Alamos National Laboratory

Dedicated to W. T. Martin

A combinatorial definition of multiple stochastic integrals is given in the setting of random measures. It is shown that some properties of such stochastic integrals, formerly known to hold in special cases, are instances of combinatorial identities on the lattice of partitions of a set. The notion of stochastic sequences of binomial type is introduced as a generalization of special polynomial sequences occuring in stochastic integration, such as Hermite, Poisson–Charlier and Kravchuk polynomials. It is shown that identities for such polynomial sets have a common origin.

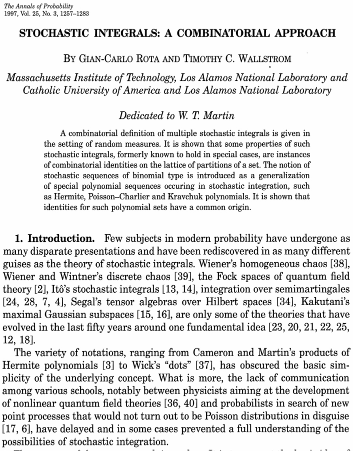
1. Introduction. Few subjects in modern probability have undergone as many disparate presentations and have been rediscovered in as many different guises as the theory of stochastic integrals. Wiener's homogeneous chaos [38], Wiener and Wintner's discrete chaos [39], the Fock spaces of quantum field theory [2], Itô's stochastic integrals [13, 14], integration over semimartingales [24, 28, 7, 4], Segal's tensor algebras over Hilbert spaces [34], Kakutani's maximal Gaussian subspaces [15, 16], are only some of the theories that have evolved in the last fifty years around one fundamental idea [23, 20, 21, 22, 25, 12, 18].

The variety of notations, ranging from Cameron and Martin's products of Hermite polynomials [3] to Wick's "dots" [37], has obscured the basic simplicity of the underlying concept. What is more, the lack of communication among various schools, notably between physicists aiming at the development of nonlinear quantum field theories [36, 40] and probabilists in search of new point processes that would not turn out to be Poisson distributions in disguise [17, 6], have delayed and in some cases prevented a full understanding of the possibilities of stochastic integration.

These asymptotic results are perfectly encoded by the **Rota-Wallstrom theory** (1997) **of combinatorial stochastic integration** — based on **Möbius calculus**.

The R-W paper is actually a **"Rosetta Stone"** for an enormous number of combinatorial formulae in the literature.

(See Roland's course for **non-commutative** versions.)



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P. A. MEYER, 1975

Université de Strasbourg
Séminaire de Probabilités 1974/75

UN COURS SUR LES INTEGRALES

STOCHASTIQUES

(Octobre 1974 / Décembre 1975)

Parmi les auditeurs du séminaire, tous mes remerciements vont à MM. G. Letta, M. Pratelli, C. Stricker, Yen Kia-An, Ch. Yoeurp pour de nombreuses corrections et améliorations. La première rédaction a été relue par Catherine Doléans - Dade, B. Maisonneuve, M. Weil et Ch. Yoeurp, qui y ont relevé d'innombrables erreurs matérielles ou mathématiques . Qu'ils trouvent ici l'expression de ma gratitude.

P.A. Meyer

sera prévisible, et l'on pourra - sous des conditions d'intégrabilité à préciser - définir

$$\int_0^\infty dX^2_{u_2} \int_0^{u_2-} f(u_1,u_2)\,dX^1_{u_1} = \int_0^\infty F(v,.)\,dX^2_v(.)$$

à condition toutefois de savoir montrer que cette intégrale stochastique ne dépend pas du choix accompli précédemment. Il reste donc beaucoup de points techniques obscurs[1]. Cependant, l'étude du choix de bonnes versions a été commencée par Catherine DOLEANS dans [20].

Maintenant, le dernier morceau : si l'on veut que la formule (41.1) puisse s'interpréter comme un résultat sur les intégrales multiples, il faut poser

$$(44.3)\qquad \int_{\{u_1=u_2\}} f(u_1,u_2)\,dX^1_{u_1}\,dX^2_{u_2} = \int_0^\infty f(v,v)\,d[X^1,X^2]_v$$

Sauf erreur de ma part, WIENER et ITO ont négligé ce terme dans leur définition de l'intégrale stochastique double par rapport au mouvement brownien. (NB : M.ZAKAI m'a dit que la méthode de WIENER en tient compte).

Passons aux intégrales d'ordre supérieur. Une intégrale triple

$$\int f(u_1,u_2,u_3)\,dX^1_{u_1}\,dX^2_{u_2}\,dX^3_{u_3}$$

se décompose en

- six intégrales du type $\int_{\{u_1<u_2<u_3\}}$, à interpréter comme des intégrales itérées.
- trois intégrales du type $\int_{\{u_1=u_2<u_3\}}$, à interpréter comme intégrales

1. Cependant, si $f(u_1,u_2)$ est une somme de produits $a(u_1)b(u_2)$, il n'y a aucune difficulté de mesurabilité, et l'on peut souvent procéder par complétion à partir de ce cas. C'est ainsi qu'on fera plus loin.

T. KAILATH & A. SEGALL, 1976

IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. IT-22, NO. 3, MAY 1976 287

Orthogonal Functionals of Independent-Increment Processes

ADRIAN SEGALL, MEMBER, IEEE, AND THOMAS KAILATH, FELLOW, IEEE

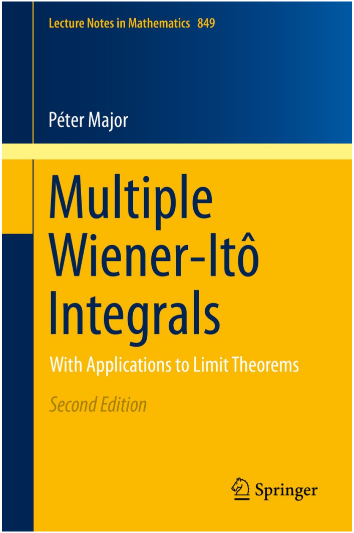
***Abstract*—In analogy with the Wiener–Itô theory of multiple integrals and orthogonal polynominals, a set of functionals of general square-integrable martingales is presented which, in the case of independent-increments processes, is orthogonal and complete in the sense that every L^2-functional of the independent-increment process can be represented as an infinite sum of these elementary functionals. The functionals are iterated integrals of the basic martingales, similar to the multiple iterated integrals of Itô and can be also thought of as being the analogs of the powers $1,x,x^2,\cdots$ of the usual calculus. The analogy is made even clearer by observing that expanding the Doleans-Dade formula for the exponential of the process in a Taylor-like series leads again to the above elementary functionals. A recursive formula for these functionals in terms of the basic martingale and of lower order functionals is given, and several connections with the theory of reproducing kernel Hilbert spaces associated with independent-increment processes are obtained.**

I. INTRODUCTION

MULTIPLE integrals of a Brownian motion (Wiener process) and the expansion of L^2-functionals in terms of these integrals were first considered by Wiener [1] and redefined in a somewhat deeper way by Itô [2]. Itô showed that his definition immediately gave mutually orthogonal terms, and he also presented their properties similar to the Hermite polynomials, but Ogura [10, footnote 6] has rightly observed that they are no longer sufficient for expanding an arbitrary L^2-nonlinear functional of a Poisson process. [The difficulty arises from several facts, one of them being that, if N_t is Poisson, $\int_0^T \alpha_t \, dN_t$ is no longer Poisson-distributed even if α_t is a deterministic function, except in very special cases as for example when $\alpha \equiv 1$. On the other hand, such integrals of a Brownian motion are always Gaussian. There are several other more important reasons that will be presented in Section III.] It, therefore, is necessary to define more general orthogonal functionals, and this is one of the aims of this paper.

After a brief description in Section II of the Wiener–Itô–Cameron–Martin expansion of L^2-functionals of Brownian motion, we define in Section III functionals of general square-integrable martingales. For processes with stationary independent increments (SII) and their (Wiener)[1] integrals, they turn out to be the appropriate generalization of the Hermite polynomials associated with the Brownian motion and its Wiener integrals. In this case, we are able to show that they also have the important property of being mutually orthogonal. The

P. Major, 1981/2014



$$h(x_1,\dots,x_{N-|\gamma|},-x_{N-2|\gamma|+1},\dots,-x_{N-|\gamma|})$$

corresponding to the diagram γ in a simple way. This yields that in the present case the function h_γ defined in (5.2) can be written in the form

$$h_\gamma(x_1,x_2,x_3,x_4) = \int\cdots\int h_1(x_1,x_5,x_6)h_2(x_7,x_8)h_3(-x_6,x_2,x_9,-x_5,-x_8)$$
$$h_4(x_3,-x_7,-x_9,x_4)G(dx_5)G(dx_6)G(dx_7)G(dx_8)G(dx_9).$$

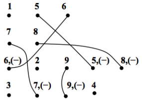


Fig. 5.3 The diagram applied for the calculation of h_γ. The sign $-$ indicates that the corresponding argument is multiplied by -1.

Here we integrate with respect to those variables x_j whose indices correspond to such a vertex of the last diagram from which an edge starts. Then the contribution of the diagram γ to the sum at the right-hand side of diagram formula equals $4!I_G(h_\gamma)$ with this function h_γ.

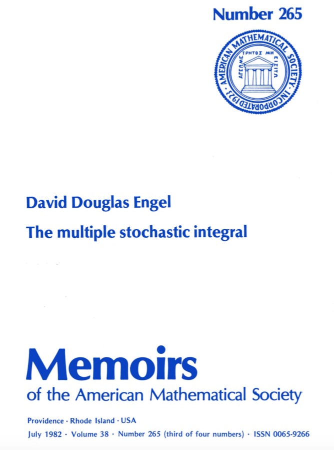
Let me remark that we had some freedom in choosing the enumeration of the vertices of the diagram γ. Thus e.g. we could have enumerated the four vertices of the diagram from which no edge starts with the numbers 1, 2, 3 and 4 in an arbitrary order. A different indexation of these vertices would lead to a different function h_γ whose Wiener–itô integral is the same. I have chosen that enumeration of the vertices which seemed to be the most natural for me.

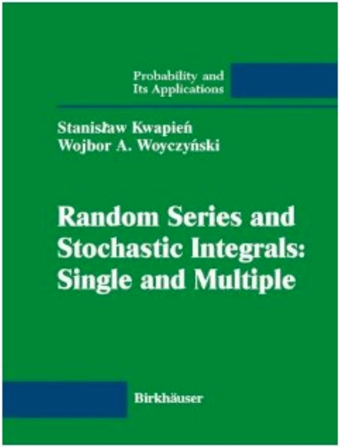
Naturally the product of two Wiener–Itô integrals can be similarly calculated, but the notation will be a bit simpler in this case. I briefly show such an example, because in the proof of Theorem 5.3 we shall be mainly interested in the product of two Wiener–Itô integrals.

Example 2. Take two Wiener–Itô integrals with kernel functions $h_1 = h_1(x_1,x_2,x_3) \in \mathcal{H}_G^3$ and $h_2 = h_2(x_1,x_2,x_3,x_4,x_5) \in \mathcal{H}_G^5$, and calculate the product $3!I_G(h_1)5!I_G(h_2)$ with the help of the diagram formula.

I shall consider only one diagram $\gamma \in \Gamma(3,5)$, and briefly explain how to calculate the kernel function h_γ of the Wiener–Itô integral corresponding to it. Let us consider for instance the diagram $\gamma \in \Gamma(3,5)$ which contains the edges $((2,1),(3,2))$ and

D. ENGEL, 1982 ; S. KWAPIEN & W. A. WOYCZYNSKI, 1992





D. SURGAILIS, 1984

ON MULTIPLE POISSON STOCHASTIC INTEGRALS AND ASSOCIATED MARKOV SEMIGROUPS

BY

D. SURGAILIS (VILNIUS)

Abstract. Multiple stochastic integrals (m.s.i.)

$$q^{(n)}(f) = \int_{X^n} f(x_1, \ldots, x_n)\, q(dx_1) \ldots q(dx_n), \qquad n = 1, 2, \ldots,$$

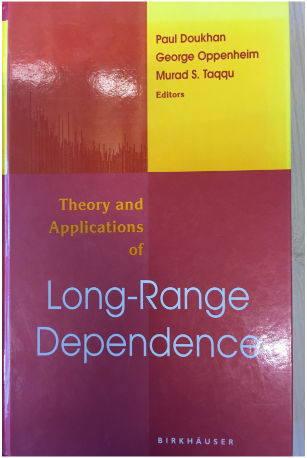
with respect to the centered Poisson random measure $q(dx)$, $\mathrm{E}[q(dx)] = 0$, $\mathrm{E}\left[(q(dx))^2\right] = m(dx)$, are discussed, where (X, m) is a measurable space. A "diagram formula" for evaluation of products of (Poisson) m.s.i. as sums of m.s.i. is derived. With a given contraction semigroup A_t, $t \geqslant 0$, in $L^2(X)$ we associate a semigroup $\Gamma(A_t)$, $t \geqslant 0$, in $L^2(\Omega)$ by the relation

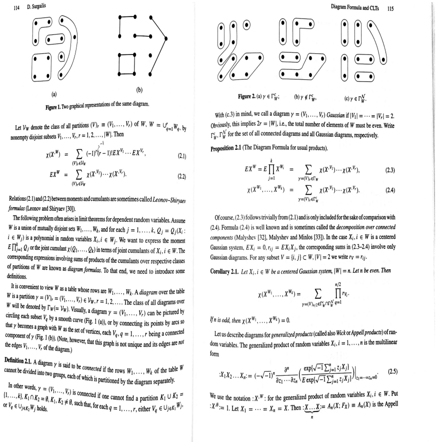
$$\Gamma(A_t)\, q^{(n)}(f_1 \hat{\otimes} \ldots \hat{\otimes} f_n) = q^{(n)}(A_t f_1 \hat{\otimes} \ldots \hat{\otimes} A_t f_n)$$

and prove that $\Gamma(A_t)$, $t \geqslant 0$, is Markov if and only if A_t, $t \geqslant 0$, is doubly sub-Markov; the corresponding Markov process can be described as time evolution (with immigration) of the (infinite) system of particles, each moving independently according to A_t, $t \geqslant 0$.

0. Introduction. It is well known that the analysis of the structure of $L^2(\Omega)$-spaces arising from the Gaussian and the Poissonian white noises has certain common features, the main one being the existence of an orthogonal system of "polynomials" ("orthogonal polynomial chaos") defined by means of multiple stochastic integrals (m.s.i.). In the Gaussian case, such integrals were first discussed by Wiener [15] and Ito [4] (on this ground called also *Wiener-Ito integrals*), and in the Poissonian case by Ito [5]. M.s.i. of both types have been applied to deal with non-linear problems in engineering (see, e.g., [16], [9], [10]), while "Gaussian" m.s.i. appeared to play a major role in many areas of mathematical physics (e.g., quantum field theory [11], statistical physics [1], [12], statistical turbulence [8], etc.). This physical interest led to a number of

Collective Work on Long-Range Dependence, 2002





114 D. Surgailis

(a) (b)

Figure 1. Two graphical representations of the same diagram.

Let $\mathcal{V}_W$ denote the class of all partitions $(V)_r \equiv (V_1, \ldots, V_r)$ of W, $W = \cup_{q=1}^r W_q$, by nonempty disjoint subsets $V_1, \ldots, V_r$, $r = 1, 2, \ldots, |W|$. Then

$$\chi(X^{\cdot W}) = \sum_{(V)_r \in \mathcal{V}_W} (-1)^r (r-1)! EX^{V_1} \cdots EX^{V_r}, \quad (2.1)$$

$$EX^W = \sum_{(V)_r \in \mathcal{V}_W} \chi(X^{\cdot V_1}) \cdots \chi(X^{\cdot V_r}). \quad (2.2)$$

Relations (2.1) and (2.2) between moments and cumulants are sometimes called *Leonov–Shiryaev formulas* (Leonov and Shiryaev [30]).

The following problem often arises in limit theorems for dependent random variables. Assume W is a union of mutually disjoint sets $W_1, \ldots, W_k$, and for each $j = 1, \ldots, k$, $Q_j = Q_j(X_i : i \in W_j)$ is a polynomial in random variables $X_i, i \in W_j$. We want to express the moment $E\prod_{j=1}^k Q_j$ or the joint cumulant $\chi(Q_1, \ldots, Q_k)$ in terms of joint cumulants of $X_i, i \in W$. The corresponding expressions involving sums of products of the cumulants over respective classes of partitions of W are known as *diagram formulas*. To that end, we need to introduce some definitions.

It is convenient to view W as a table whose rows are $W_1, \ldots, W_k$. A *diagram* over the table W is a partition $\gamma = (V)_r = (V_1, \ldots, V_r) \in \mathcal{V}_W$, $r = 1, 2, \ldots$. The class of all diagrams over W will be denoted by $\Gamma_W (= \mathcal{V}_W)$. Visually, a diagram $\gamma = (V_1, \ldots, V_r)$ can be pictured by circling each subset V_q by a smooth curve (Fig. 1 (a)), or by connecting its points by arcs so that γ becomes a graph with W as the set of vertices, each $V_q, q = 1, \ldots, r$ being a connected component of γ (Fig. 1 (b)). (Note, however, that this graph is not unique and its edges are *not* the edges $V_1, \ldots, V_r$ of the diagram.)

Definition 2.1. A diagram γ is said to be *connected* if the rows $W_1, \ldots, W_k$ of the table W cannot be divided into two groups, each of which is partitioned by the diagram separately.

In other words, $\gamma = (V_1, \ldots, V_r)$ is connected if one cannot find a partition $K_1 \cup K_2 = \{1, \ldots, k\}$, $K_1 \cap K_2 = \emptyset$, $K_1, K_2 \neq \emptyset$, such that, for each $q = 1, \ldots, r$, either $V_q \in \cup_{j \in K_1} W_j$, or $V_q \in \cup_{j \in K_2} W_j$ holds.

Diagram Formula and CLTs 115

Figure 2. (a) $\gamma \in \Gamma_W^c$; (b) $\gamma \notin \Gamma_W^c$, (c) $\gamma \in \Gamma_W^{\mathcal{N}}$.

With (c.3) in mind, we call a diagram $\gamma = (V_1, \ldots, V_r)$ *Gaussian* if $|V_1| = \cdots = |V_r| = 2$. Obviously, this implies $2r = |W|$, i.e., the total number of elements of W must be even. Write $\Gamma_W^c, \Gamma_W^{\mathcal{N}}$ for the set of all connected diagrams and all Gaussian diagrams, respectively.

Proposition 2.1 (The Diagram Formula for usual products).

$$EX^W = E\prod_{j=1}^k X^{W_i} = \sum_{\gamma=(V)_r \in \Gamma_W} \chi(X^{\cdot V_1}) \cdots \chi(X^{\cdot V_r}), \quad (2.3)$$

$$\chi(X^{W_1}, \ldots, X^{W_k}) = \sum_{\gamma=(V)_r \in \Gamma_W^c} \chi(X^{\cdot V_1}) \cdots \chi(X^{\cdot V_r}). \quad (2.4)$$

Of course, (2.3) follows trivially from (2.1) and is only included for the sake of comparison with (2.4). Formula (2.4) is well known and is sometimes called the *decomposition over connected components* (Malyshev [32], Malyshev and Minlos [33]). In the case $X_i, i \in W$ is a centered Gaussian system, $EX_i = 0, r_{ij} = EX_iX_j$, the corresponding sums in (2.3–2.4) involve only Gaussian diagrams. For any subset $V = \{i, j\} \subset W, |V| = 2$ we write $r_V = r_{ij}$.

Corollary 2.1. *Let $X_i, i \in W$ be a centered Gaussian system, $|W| = n$. Let n be even. Then*

$$\chi(X^{W_1}, \ldots, X^{W_k}) = \sum_{\gamma=(V)_{n/2} \in \Gamma_W^c \cap \Gamma_W^{\mathcal{N}}} \prod_{q=1}^{n/2} r_{V_i}.$$

If n is odd, then $\chi(X^{W_1}, \ldots, X^{W_k}) = 0$.

Let us describe diagrams for *generalized products* (called also *Wick* or *Appell products*) of random variables. The generalized product of random variables $X_i, i = 1, \ldots, n$ is the multilinear form

$$:X_1X_2\ldots X_n: = (-\sqrt{-1})^n \frac{\partial^n}{\partial z_1 \cdots \partial z_n} \left(\frac{\exp\{\sqrt{-1}\sum_{j=1}^n z_jX_j\}}{E\exp\{\sqrt{-1}\sum_{j=1}^n z_jX_j\}} \right) \Big|_{z_1=\cdots=z_n=0}. \quad (2.5)$$

We use the notation $:X^{\cdot W}:$ for the generalized product of random variables $X_i, i \in W$. Put $:X^{\cdot \emptyset}: = 1$. Let $X_1 = \cdots = X_n = X$. Then $:\underbrace{X\ldots X}_n: = A_n(X; F_X) \equiv A_n(X)$ is the Appell

M. Sodin and B. Tsirelson, 2004

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RANDOM COMPLEX ZEROES, I.
ASYMPTOTIC NORMALITY

BY

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ABSTRACT

We consider three models (elliptic, flat and hyperbolic) of Gaussian random analytic functions distinguished by invariance of their zeroes distribution. Asymptotic normality is proven for smooth functionals (linear statistics) of the set of zeroes.

Introduction and the main result

Zeroes of random polynomials and other analytic functions were studied by mathematicians and physicists under various assumptions on random coefficients. One class of models introduced not long ago by Bogomolny, Bohigas and Leboeuf [5, 6], Kostlan [16], and Shub and Smale [23] has a remarkably unique unitary invariance:

> "...indeed it has no true freedom at all. It is (statistically) unique in the same sense as 'the Poisson process', or 'the thermal (black body) electromagnetic field' are unique..."
> Hannay [13, p. L755]

Following Hannay [12], we use the term 'chaotic analytic zero points' (CAZP, for short). We consider here three CAZP models: the elliptic CAZP, the flat CAZP, and the hyperbolic CAZP called by Leboeuf [17, p. 654] $SU(2)$, W_1, and

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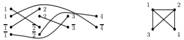
Proof of (2.16): First, we make another reduction of the diagram and define a p-vertex graph with simple edges which couple the vertices i and j if and only if at least one of the pairs $(i,\bar{j})$ or $(\bar{j},i)$ was coupled in the original diagrams (without taking into account the multiplicities of the original coupling). We denote the reduced diagram by γ^{**}. For example:



a diagram, γ its reduction, γ^{**}

Then

$$|V_\gamma(t_1,\dots,t_p)| \le \prod_{(i,j)\in e(\gamma^{**})} |\rho(t_i,t_j)| \tag{2.17}$$

where the product is taken over all edges of γ^{**}. We have to estimate from above the integral of $|V_\gamma|$ over T^p. Replacing $|V_\gamma|$ by its upper bound (2.17), we obtain the integral which factorizes into the product of integrals described by connected components of the diagram γ^{**}.

Let us start with one m-vertex component of the diagram γ^{**}. The component can be a complicated graph — anyway, we can always turn this graph into a tree with m vertices by deleting some edges (this procedure only increases the integral we are estimating). Having a tree, we choose a vertex belonging to only one edge and integrate it out, which gives the factor $(\sup_{s\in T}\int_T|\rho_n(s,t)|d\mu(t))$ and the rest of the tree which is a new tree with $m-1$ vertices. By induction, any m-vertex tree describes the integral which does not exceed

$$\left(\sup_{s\in T}\int_T |\rho_n(s,t)|d\mu(t)\right)^{m-1}.$$

Now, suppose the reduced diagram γ^{**} has k connected components and the i-th component has m_i vertices.[10] Then the right-hand side of (2.17) integrated over T^p does not exceed

$$\left(\sup_{s\in T}\int_T |\rho_n(s,t)|d\mu(t)\right)^{(m_1-1)+\cdots+(m_k-1)} = \left(\sup_{s\in T}\int_T |\rho_n(s,t)|d\mu(t)\right)^{p-k}.$$

Since the diagram γ is irregular, $k<p/2$ and we get (2.16).

10 Observe that $m_1+\cdots+m_k=p$.

D. Marinucci, 2007

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A central limit theorem and higher order results for the angular bispectrum

Domenico Marinucci

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Abstract The angular bispectrum of spherical random fields has recently gained an enormous importance, especially in connection with statistical inference on cosmological data. In this paper, we analyze its moments and cumulants of arbitrary order and we use these results to establish a multivariate central limit theorem and higher order approximations. The results rely upon combinatorial methods from graph theory and a detailed investigation for the asymptotic behavior of coefficients arising in matrix representation theory for the group of rotations $SO(3)$.

Keywords Spherical random fields · Angular bispectrum · Central limit theorem · Higher order approximations

Mathematics Subject Classification (2000) Primary: 60G60; Secondary: 60F05 · 62M15 · 62M40

1 Introduction

Let $T(\theta, \varphi)$ be a random field indexed by the unit sphere S^2, $0 \leq \theta \leq \pi$ and $0 \leq \varphi < 2\pi$. We assume that $T(\theta, \varphi)$ has zero mean, finite variance and it is mean square continuous and isotropic, i.e., its covariance is invariant with respect to the group of

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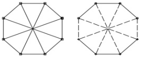
Fig. 1 The partition of a graph into two subgraphs



$X_{R_i;\gamma_i}$ can be viewed as a vector whose elements are indexed by m_{r_i,k_i}, where $r_i \in R_i$ and $\{(r_i,k_i),.\} \notin \gamma_i$ (indeed those indexes $m_{r_ik_i}$ such that $\{(r_i,k_i),.\} \in \gamma_i$ have been summed up internally). For instance, for $g = 2$ we have

$$D(\gamma) = \sum_{\substack{m_{rk}=-\ell_{rk} \\ \{(r,k),.\}\in\gamma_{12}}}^{\ell_{rk}} \left\{ \sum_{\substack{m_{rk}=-\ell_{rk} \\ \{(r,k),.\}\in\gamma_1}}^{\ell_{rk}} \prod_{r\in R_1} \begin{pmatrix} \ell_{r1} & \ell_{r2} & \ell_{r3} \\ m_{r1} & m_{r2} & m_{r3} \end{pmatrix} \delta(\gamma_1) \right\}$$
$$\times \left\{ \sum_{\substack{m_{rk}=-\ell_{rk} \\ \{(r,k),.\}\in\gamma_2}}^{\ell_{rk}} \prod_{r\in R_2} \begin{pmatrix} \ell_{r1} & \ell_{r2} & \ell_{r3} \\ m_{r1} & m_{r2} & m_{r3} \end{pmatrix} \delta(\gamma_2) \right\} \delta(\gamma_{12}).$$

In Fig. 1, we provide a graph with eight nodes $\#(R) = 8$ (right), and then (left) we partition it with $g = 2$, $\#(R_1) = \#(R_2) = 4$; the nodes in R_1 are labelled with a circle, the nodes in R_2 are labelled with a cross, the edges in γ_1 and γ_2 have a solid line while those in γ_{12} are dashed. Here we have $3 + 3 = 6$ internal sums and six external ones.

Assume now that γ_i does not include any loop, for $i = 1, \ldots, g$. We shall show that

$$|D(\gamma)| \leq \prod_{i=1}^{g} \|X_{R_i;\gamma_i}\| \tag{11}$$

where $\|.\|$ denotes Euclidean norm, and

$$\|X_{R_i;\gamma_i}\| \leq \prod_{\{(r,k),.\}\in\gamma_i} (2\ell_{rk}+1)^{-1/2} \leq (2 \min_{\{(r,k),.\}\in\gamma_i} \ell_{rk} + 1)^{(\#(R_i)-1)/2}; \tag{12}$$

note that if γ_i does not include any loop the number of edges it contains must be identically equal to $\#(R_i) - 1$, where $\#(.)$ denotes the cardinality of a set. Let us consider (11) first. It is clear that we can choose new indexes such that $X_{R_i;\gamma_i} =: X^{(i)}$ is a vector with elements

$$X^{(i)} = \left\{ X^{(i)}_{m_{i1},\ldots,m_{iv_i}}, -\ell_{ij} \leq m_{ij} \leq \ell_{ij}, j = 1, \ldots v_i \right\}, \quad i = 1, \ldots, g;$$

J. Buckley and M. Sodin, 2017

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Fluctuations of the Increment of the Argument for the Gaussian Entire Function

Jeremiah Buckley[1] · Mikhail Sodin[2]

Received: 22 January 2017 / Accepted: 12 May 2017

Abstract The Gaussian entire function is a random entire function, characterised by a certain invariance with respect to isometries of the plane. We study the fluctuations of the increment of the argument of the Gaussian entire function along planar curves. We introduce an inner product on finite formal linear combinations of curves (with real coefficients), that we call the signed length, which describes the limiting covariance of the increment. We also establish asymptotic normality of fluctuations.

Keywords Zeroes of holomorphic functions · Gaussian processes · Point processes

Let $(\zeta_n)_{n=0}^{\infty}$ be a sequence of iid standard complex Gaussian random variables (that is, each ζ_n has density $\frac{1}{\pi}e^{-|z|^2}$ with respect to the Lebesgue measure on the plane), and define the Gaussian entire function by

$$f(z) = \sum_{n=0}^{\infty} \zeta_n \frac{z^n}{\sqrt{n!}}. \qquad (1)$$

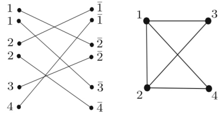


Fig. 6 A diagram and its reduced diagram

We now fix γ and $\mathfrak{P}_{P-2\gamma}$ and make a reduction to allow us to estimate this quantity. From the irregular diagram $\mathcal{D}$ we form the reduced diagram $\mathcal{D}^*$ (see Fig. 6) with P vertices (labelled 1 to P) such that:

- For each $1 \le r, s \le P$ there is at most one edge (r, s).
- $(r, s) \in e(\mathcal{D}^*)$ if $(r, \bar{s}) \in e(\mathcal{D})$ or $(s, \bar{r}) \in e(\mathcal{D})$.

In other words we form $\mathcal{D}^*$ from $\mathcal{D}$ by glueing together the $2\alpha_r$ vertices labelled r or $\bar{r}$ for each r, and ignoring the multiplicity of the edges of the resultant diagram. We decompose

$$\mathcal{D}^* = \bigcup_{u=1}^{n} \mathcal{D}_u$$

into n connected components that contain a_u vertices and contribute ℓ_u factors to $\mathfrak{P}_{P-2\gamma}$. Notice that $n < \frac{P}{2}$ since $\mathcal{D}$ is irregular, and that

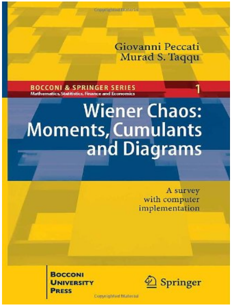
$$\sum_{u=1}^{n} a_u = P \quad \text{and} \quad \sum_{u=1}^{n} \ell_u = P - 2\gamma.$$

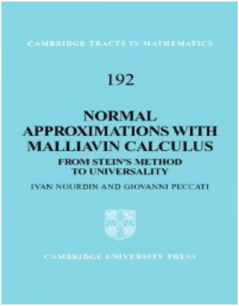
Moreover, since

$$|V(\mathcal{D})| = \prod_{(r,\bar{s})\in e(\mathcal{D})} \exp\{-\tfrac{R^2}{2}|z_r - z_s|^2\} \le \prod_{(r,s)\in e(\mathcal{D}^*)} \exp\{-\tfrac{R^2}{2}|z_r - z_s|^2\}$$

A Note on Coauthors

- ⋆ My view of the Rota-Wallstrom theory has formed while working with **M.S. Taqqu** (Boston).
- ⋆ The approach to probabilistic approximations via variational tools has been principally developed in collaboration with **I. Nourdin** (Luxembourg).





The Plan

1. Lattice of partitions and elements of Möbius calculus.
2. Cumulants
3. Random measures and the Rota-Wallstrom Theory
4. Product and diagram formulae
5. Limit theorems
6. Geometric application (blackboard/handout)

From now on: everything random lives on an adequate triple

$$''(\Omega, \mathcal{F}, \mathbb{P})'' \quad (\mathbb{E} = \text{expectation})$$

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PART 1:
LATTICE OF PARTITIONS AND MÖBIUS CALCULUS

PARTITIONS

- Given $n \geq 1$, we write $[n] = \{1, ..., n\}$. $\mathcal{P}_n$ is the **poset of partitions** of $[n]$, with partial order relation (by inclusion of blocks) denoted by $\preccurlyeq$. A partition

$$\pi = \{b_1, ..., b_r\} \in \mathcal{P}_n$$

has $|\pi| := r$ **blocks**. If $b \subseteq [n]$, write $\mathcal{P}(b)$:= poset of partitions of b.

- Write $i \sim_\pi j$ if i, j are in the same block of π.

- The **minimal** and **maximal** partitions of $\mathcal{P}_n$ are, respectively,

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EXAMPLE OF $\mathcal{P}_3$

- ⋆ There are only five partitions: $\hat{1}$, $\hat{0}$,

  $\pi_1 = \{\{1\},\{1,2\}\}$, $\pi_2 = \{\{1,3\},\{2\}\}$, $\pi_3 = \{\{1,2\},\{3\}\}$.

- ⋆ One has

$$\hat{0} \preccurlyeq \pi_i \preccurlyeq \hat{1}, \quad i = 1,2,3.$$

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$$\pi_i \not\preccurlyeq \pi_j, \quad i \neq j.$$

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Lattice Structure

- ⋆ The poset $\mathcal{P}_n$ is actually a **lattice**, since one can define two operations of **meet** and **join**, from $\mathcal{P}_n \times \mathcal{P}_n$ onto $\mathcal{P}_n$, written respectively

$$\begin{array}{rcl}(\sigma, \pi) & \mapsto & \sigma \wedge \pi \\ (\sigma, \pi) & \mapsto & \sigma \vee \pi.\end{array}$$

- ⋆ The meet $\sigma \wedge \pi$ is uniquely characterized by the properties: **(i)** $\sigma \wedge \pi \preccurlyeq \sigma, \pi$, and **(ii)** if $\gamma \preccurlyeq \sigma, \pi$, then $\gamma \preccurlyeq \sigma \wedge \pi$.
- ⋆ The join $\sigma \vee \pi$ is uniquely characterized by the properties: **(i)** $\sigma, \pi \preccurlyeq \sigma \vee \pi$, and **(ii)** if $\sigma, \pi \preccurlyeq \gamma$, then $\sigma \vee \pi \preccurlyeq \gamma$.

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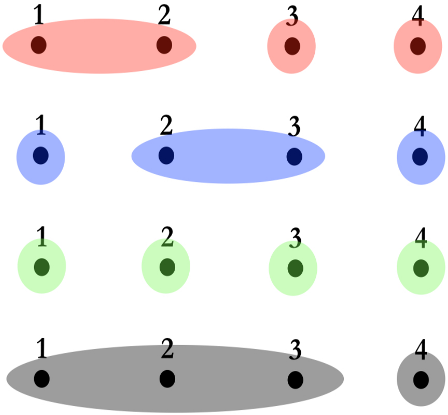
Meet and Join



$\sigma = \{\{1,2\},\{3\},\{4\}\}$

$\pi = \{\{1\},\{2,3\},\{4\}\}$

$\sigma \wedge \pi = \hat{0}$

$\sigma \vee \pi = \{\{1,2,3\},\{4\}\}$

SEGMENTS

- The **segment** associated with two partitions $\sigma \preccurlyeq \pi$ is

$$[\sigma, \pi] := \{\rho \in \mathcal{P}_n : \sigma \preccurlyeq \rho \preccurlyeq \pi\},$$

in such a way that $[\hat{0}, \hat{1}] = \mathcal{P}_n$.

- The **class** $\lambda(\sigma, \pi)$ of the segment $[\sigma, \pi]$ is the formal string

$$\lambda(\sigma, \pi) := (1^{r_1} 2^{r_2} \cdots |\sigma|^{r_{|\sigma|}}),$$

indicating that π has exactly r_i blocks containing exactly i blocks of σ.

- $\lambda(\sigma, \pi)$ can be regarded as a **partition of the integer** $|\sigma|$:

$$1r_1 + 2r_2 + \cdots + r_{|\sigma|}|\sigma| = |\sigma|, \quad r_1 + \cdots + r_{|\sigma|} = |\pi|.$$

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- $\lambda(\sigma, \hat{1}) = (1^0 2^0 \cdots |\sigma|^1)$.
- $\lambda(\hat{0}, \sigma) = (1^{r_1} 2^{r_2} \cdots n^{r_n})$, where $r_i :=$ # blocks of σ of size i.
- Case $n = 5$,

$$\sigma = \{\{1\},\{2\},\{3\},\{4,5\}\},\ \pi = \{\{1\},\{2,3,4,5\}\}.$$

Then,

$$[\sigma, \pi] = \{\sigma\,;\ \{\{1\},\{2\},\{3,4,5\}\}\,;\ \{\{1\},\{2,4,5\},\{3\}\}\,;\ \{\{1\},\{2,3\},\{4,5\}\}\,;\ \pi\}$$

and $\lambda(\sigma, \pi) = (1^1 2^0 3^1 4^0)$.

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Möbius Functions, I

- ⋆ The **incidence algebra** of $\mathcal{P}_n$ is the class of all functions $f : \mathcal{P}_n \times \mathcal{P}_n \to \mathbb{C}$ such that

$$f(\sigma, \pi) = f(\sigma, \pi)\mathbf{1}_{\{\sigma \preccurlyeq \pi\}}.$$

- ⋆ Two distinguished elements of $\mathcal{I}_n$ (**zeta function** and **identity** for $\star$):

$$\zeta(\sigma, \pi) = \mathbf{1}_{\{\sigma \preccurlyeq \pi\}}, \quad \text{and} \quad \delta(\sigma, \pi) = \mathbf{1}_{\{\sigma = \pi\}}.$$

- ⋆ **Convolution** on the incidence algebra $\mathcal{I}_n$ of $\mathcal{P}_n$ is

$$f \star g(\sigma, \pi) = \sum_{\varrho \in [\sigma, \pi]} f(\sigma, \varrho) g(\varrho, \pi),$$

with $\Sigma_{\emptyset} \equiv 0$.

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⋆ The associated **Möbius function** [i.e., the two-sided inverse of ζ on $(\mathcal{I}_n, \star)$] is denoted by $\mu(\cdot,\cdot)$: it is characerized by the relations: (a) $\mu(\pi,\pi) = 1$, (b)

$$\mu(\sigma,\pi) = -\sum_{\sigma \preccurlyeq \varrho \prec \pi} \mu(\sigma,\varrho) = -\sum_{\sigma \prec \varrho \preccurlyeq \pi} \mu(\varrho,\pi), \quad \sigma \prec \pi,$$

and (c) $\mu(\sigma,\pi) = 0$ otherwise.

⋆ For partitions $\sigma \preccurlyeq \pi$ such that $m = |\sigma| \geq |\pi| = r$, the function μ is explicitly given by

$$\boxed{\mu(\sigma,\pi) = (-1)^{m-r}(2!)^{r_3}(3!)^{r_4}\cdots((m-1)!)^{r_m}}$$

where $\lambda(\sigma,\pi) = (1^{r_1}2^{r_2}\cdots m^{r_m})$ is the class of $[\sigma,\pi]$. In particular, $\mu(\pi,\pi) = 1$, and $\boxed{\mu(\sigma,\hat{1}) = (-1)^{|\sigma|-1}(|\sigma|-1)!}$

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- ⋆ For partitions $\sigma \preccurlyeq \pi$ such that $m = |\sigma| \geq |\pi| = r$, the function μ is explicitly given by

$$\boxed{\mu(\sigma,\pi) = (-1)^{m-r}(2!)^{r_3}(3!)^{r_4}\cdots((m-1)!)^{r_m}}$$

  where $\lambda(\sigma,\pi) = (1^{r_1}2^{r_2}\cdots m^{r_m})$ is the class of $[\sigma,\pi]$. In particular, $\mu(\pi,\pi) = 1$, and $\boxed{\mu(\sigma,\hat{1}) = (-1)^{|\sigma|-1}(|\sigma|-1)!}$

Möbius Function, III

A small number of (elementary) properties of μ are used below:

(i) **Inversion:**

$$F(\pi) = \sum_{\sigma \preccurlyeq \pi} G(\sigma) \text{ if and only if } G(\pi) = \sum_{\sigma \preccurlyeq \pi} \mu(\sigma, \pi) F(\sigma)$$

$$F(\pi) = \sum_{\sigma \succcurlyeq \pi} G(\sigma) \text{ if and only if } G(\pi) = \sum_{\sigma \succcurlyeq \pi} \mu(\pi, \sigma) F(\sigma)$$

(ii) **On segments** :

$$\sum_{\varrho \in [\sigma, \pi]} \mu(\varrho, \pi) = \sum_{\varrho \in [\sigma, \pi]} \mu(\sigma, \varrho) = \delta(\sigma, \pi).$$

(iii) Möbius functions are **preserved by isomorphisms**.

(iv) Möbius functions **factorize on lattice products**.

Flat and Connected Diagrams

- ⋆ Consider a partition with the form

$$\pi^\star = \Big\{\{1,...,n_1\},\{n_1+1,...,n_1+n_2\},\\ ....,\{n_1+\cdots+n_{k-1}+1,...,n_1+\cdots+n_k\}\Big\}$$

$(n_1+\cdots+n_k=n)$

- ⋆ Solutions to the equation $\sigma\wedge\pi^\star=\hat{0}$ have a representation in terms of **non-flat diagrams**.
- ⋆ Solutions to the equation $\sigma\vee\pi^\star=\hat{1}$ have a representation in terms of **connected diagrams**.

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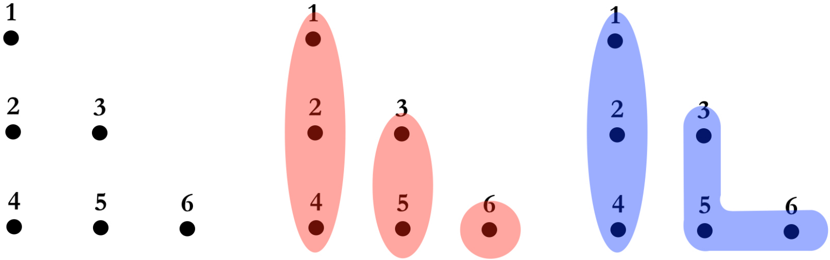
Non-Flat vs. Flat



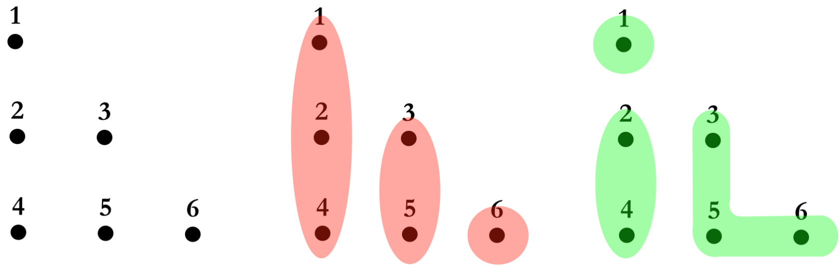
$\pi^\star = \{\{1\},\{2,3\},\{4,5,6\}\}$

$\sigma = \{\{1,2,4\},\{3,5\},\{6\}\}$, $\varrho = \{\{1,2,4\},\{3,5,6\}\}$

$\pi^\star \wedge \sigma = \hat{0}$

$\pi^\star \wedge \varrho = \{\{1\},\{2\},\{3\},\{4\},\{5,6\}\} \neq \hat{0}$

Connected vs. Non-Connected



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$$\sigma = \{\{1,2,4\},\{3,5\},\{6\}\}, \quad \gamma = \{\{1\},\{2,4\},\{3,5,6\}\}$$

$$\pi^\star \vee \sigma = \hat{1}$$

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PART 2:
CUMULANTS

Notation and Setting

- ⋆ For a fixed $n \geq 1$, we consider a vector of r.v.'s $\mathbf{X}_{[n]} = (X_1, ..., X_n)$ s.t. $\mathbb{E}|X_j|^n < \infty$, $j = 1, ..., n$.

- ⋆ For every $b = \{j_1, ..., j_k\} \subseteq [n]$,

$$\mathbf{X}_b := (X_{j_1}, ..., X_{j_k}), \quad \mathbf{X}^b := X_{j_1} \times \cdots \times X_{j_k},$$

and

$$g_{\mathbf{X}_b}(t_1, ..., t_k) := \mathbb{E}\left[\exp\left\{i \sum_{\ell=1}^{k} t_\ell X_{j_\ell}\right\}\right].$$

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Cumulants

- ⋆ Define the **cumulant** $\chi(\mathbf{X}_b)$ of $\mathbf{X}_b$ as

$$\chi(\mathbf{X}_b) := (-i)^k \frac{\partial^k}{\partial t_1 \cdots \partial t_k} \log g_{\mathbf{X}_b}(t_1, ..., t_k) \mid_{t_1 = \cdots = t_k = 0} .$$

- ⋆ For a single random variable X s.t. $\mathbb{E}|X|^n < \infty$, the n**th cumulant** is

$$\chi_n(X) := \chi(X, ..., X) := (-i)^n \frac{\partial^n}{\partial z^n} \log g_X(z) \mid_{z=0},$$

and consequently $\chi_1(X) = \mathbb{E}(X)$.

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First Properties

- ⋆ The mapping $\mathbf{X}_b \mapsto \chi(\mathbf{X}_b)$ is **homogeneous**:

$$\chi\left(v_1 X_{j_1}, ..., v_k X_{j_k}\right) = v_1 \cdots v_k \, \chi(\mathbf{X}_b);$$

- ⋆ the application $\mathbf{X}_b \mapsto \chi(\mathbf{X}_b)$ is **invariant with respect to permutations** of b;
- ⋆ if the vector $\mathbf{X}_b$ has the form $\mathbf{X}_b = \mathbf{X}_{b'} \cup \mathbf{X}_{b''}$, with $b', b'' \neq \varnothing$, $b' \cap b'' = \varnothing$ and $\mathbf{X}_{b'}$ and $\mathbf{X}_{b''}$ independent, then $\chi(\mathbf{X}_b) = 0$;
- ⋆ if $\mathbf{Y} = \{Y_i : i \in I\}$ is a Gaussian family and if $\mathbf{X}$ is a vector obtained by juxtaposing $n \geq 3$ elements of $\mathbf{Y}$ (with possible repetitions), then $\chi(\mathbf{X}) = 0$.

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Leonov-Shiryaev Relations

Theorem (Leonov & Shiryaev, 1959)

For every $b \subseteq [n]$,

1. $$\mathbb{E}(\mathbf{X}^b) = \sum_{\pi=\{b_1,\ldots,b_k\}\in\mathcal{P}(b)} \chi(\mathbf{X}_{b_k})\cdots\chi(\mathbf{X}_{b_1});$$

2. $$\chi(\mathbf{X}_b) = \sum_{\sigma=\{a_1,\ldots,a_r\}\in\mathcal{P}(b)} (-1)^{r-1}(r-1)!\mathbb{E}(\mathbf{X}^{a_1})\cdots\mathbb{E}(\mathbf{X}^{a_k}).$$

Remark. In the case $\mathbf{X}_b = (X,\ldots,X)$ (n times):

$$\boxed{\mathbb{E}(X^n) = \textstyle\sum_{\sigma=\{b_1,\ldots,b_j\}\in\mathcal{P}_n} \chi_{|b_1|}(X)\times\cdots\times\chi_{|b_j|}(X)}.$$

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Sketch of Proof

By Leibniz formula,

$$\begin{aligned}
&\mathbb{E}(X^n)\\
&= (-i)^n D^n g_X(0)\\
&= (-i)^n D^{n-1}\left(D \log g_X \cdot g_X\right)(0)\\
&= \sum_{s=0}^{n-1} \binom{n-1}{s} (-1)^{s+1} D^{s+1} \log g_X(0)\\
&\qquad\qquad \times (-i)^{n-s-1} D^{n-s-1} \log g_X(0)\\
&= \sum_{s=0}^{n-1} \binom{n-1}{s} \chi_{s+1}(X)\, \mathbb{E}\left(X^{n-s-1}\right), \text{ and then use recursion.}
\end{aligned}$$

Point 2 is Möbius inversion ! □

Examples

- $$\chi(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = \mathbf{Cov}(X,Y)$$

- $$\chi_3(X) = 2\mathbb{E}(X)^3 - 3\mathbb{E}(X)\mathbb{E}(X^2) + \mathbb{E}(X^3)$$

- $$\chi_4(X) = -6\mathbb{E}(X)^4 + 12\mathbb{E}(X)^2\mathbb{E}(X^2) - 3\mathbb{E}(X^2)^2 - 4\mathbb{E}(X)\mathbb{E}(X^3) + \mathbb{E}(X^4)$$

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Poisson, Bell and Touchard

- Let Π be a **Poisson random variable** with parameter 1 (that is, $\mathbb{P}[\Pi = k] = e^{-1}/k!$). Then, $g_{\Pi}(t) = \exp\left[e^{it} - 1\right]$, so that $\chi_n(\Pi) = 1, n \geq 1$.
- It follows that

$$\mathbb{E}(\Pi^n) = |\mathcal{P}_n| = B_n \text{ (the } n\text{th } \textbf{Bell number}),$$

- This is equivalent to the "Dobinsky formula" (1887)

$$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}, \quad n \geq 1.$$

- Taking an arbitrary parameter $\lambda > 0$ generates the **Touchard polynomials**, whose coefficients are the Stirling numbers of the second kind. Also,

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Feynman Formula

- $\star$ Let $(G_1, ..., G_d)$ be a centered Gaussian vector with covariance matrix $\{C(i,j) : i,j = 1, ..., d\}$.

- $\star$ One has that $\mathbb{E}[G_1 \cdots G_d] = 0$ if d is odd, and

$$\mathbb{E}[G_1 \cdots G_d] = \sum_{\{\{i_1,j_1\},...,\{i_{d/2},j_{d/2}\}\} \in \mathcal{M}_d} \prod_{\ell=1}^{d/2} C(i_\ell, j_\ell),$$

where $\mathcal{M}_d$ is the collection of all **matchings** of $[d]$ (a matching is a partition with blocks of size 2).

- $\star$ In particular, if $G \sim \mathcal{N}(0, \sigma^2)$,

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Leonov and Shiryaev, 1959

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SHORT COMMUNICATIONS

ON A METHOD OF CALCULATION OF SEMI-INVARIANTS

V. P. LEONOV and A. N. SHIRYAEV

(Translated by James R. Brown)

Methods of calculation of leading moments and semi-invariants in the study of non-linear transformations of random processes are rather complicated (cf., for instance, [1], [2]). [3] contains the interesting remark that in some cases, after determining the dependence of the moments m_η of the polynomial $\eta = Q(\xi)$ on the semi-invariants s_ξ of the process $\xi(t)$, the semi-invariants of the process $\eta(t)$ are found by simply cancelling the superfluous terms according to a determined rule. Our theorem in §2 shows that this remark continues to hold even under considerably more general assumptions.

The present work was carried out under the direction of A. N. Kolmogorov, to whom the authors express their gratitude for posing the problem as well as for numerous valuable directions received in the course of its solution.

1. Introduction

Let a random vector $\eta = \{\eta_1, \cdots, \eta_k\}$ be given. Let us look at its characteristic function

$$\varphi_\eta(\alpha_1, \cdots, \alpha_k) = \mathbf{M} e^{i(\alpha_1 \eta_1 + \cdots + \alpha_k \eta_k)}.$$

We shall assume that $\mathbf{M}|\eta_j|^n < \infty$.T-1 Then the mixed moments $\mathbf{M}\{\eta_1^{\nu_1} \cdots \eta_k^{\nu_k}\}$ exist for all $\nu_1, \cdots, \nu_k$ such that $\nu_j \geqq 0$, $\nu_1 + \cdots + \nu_k \leqq n$. It follows that the partial derivatives

$$\frac{\partial^{\nu_1 + \cdots + \nu_k}}{\partial \alpha_1^{\nu_1} \cdots \partial \alpha_k^{\nu_k}} \varphi_\eta(\alpha_1, \cdots, \alpha_k)$$

We shall calculate s_η by formula (IV.d). In the example we are looking at the set $\mathfrak{D}$ consists of the pairs of integers (i, j) such that $1 \leqq i \leqq k$, $j = 1, 2$; table (A) has the form

$$\begin{matrix} (1, 1) & (1, 2) \\ (2, 1) & (2, 2) \\ \cdots & \cdots \\ (k, 1) & (k, 2) \end{matrix}$$

$a_\nu = 1$, if $\nu = \chi(\mathfrak{D})$ and $a_\nu = 0$ if $\nu \neq \chi(\mathfrak{D})$. Formula (IV.d) assumes the form

$$s_\eta = \sum_{\bigcup_{p=1}^{q} \mathfrak{D}_p = \mathfrak{D}}^{*} \prod_{p=1}^{q} s_\xi(\mathfrak{D}_p).$$

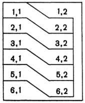
Since ξ is a vector with gaussian distribution, $s_\xi(\mathfrak{D}_p) \neq 0$ only when $\mathfrak{D}_p$ consists of one or two points.

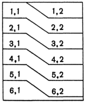
Let e_i and $\bar{e}_i$ equal either 1 or 2, while $e_i \neq \bar{e}_i$. Using Remark 5 we may show that the partition $\mathfrak{D} = \bigcup_{p=1}^{q} \mathfrak{D}_p$ is indecomposable if and only if one of the following two cases holds:

1. $$\mathfrak{D}_1 = \{(i_1 \bar{e}_1), (i_2, e_2)\}, \quad \mathfrak{D}_2 = \{(i_2, \bar{e}_2), (i_3, e_3)\}, \cdots, \\ \mathfrak{D}_{k-1} = \{(i_{k-1}, \bar{e}_{k-1}), (i_k, e_k)\}, \quad \mathfrak{D}_k = \{(i_k, e_k), (i_1, e_1)\},$$

2. $$\mathfrak{D}_1 = \{(i_1, e_1)\}, \quad \mathfrak{D}_2 = \{(i_1, \bar{e}_1), (i_2, e_2)\}, \quad \mathfrak{D}_3 = \{(i_2, \bar{e}_2), (i_3, e_3)\}, \cdots, \\ \mathfrak{D}_k = \{(i_{k-1}, \bar{e}_{k-1}), (i_k, e_k)\}, \quad \mathfrak{D}_{k+1} = \{(i_k, \bar{e}_k)\},$$

where $(i_1, i_2, \cdots, i_k)$ is some permutation of the numbers $1, 2, \cdots, k$. For the case $k = 6$, $i_p = p$, $\bar{e}_p = 1$, $e_p = 2$ these partitions are illustrated in Figures 1 and 2, respectively.





T. Speed, 1983 ; G.-C. Rota and J. Shen, 2000

Austral. J. Statist., **25** (**2**), 1983, 378–388

CUMULANTS AND PARTITION LATTICES[1]

T. P. Speed
CSIRO Division of Mathematics and Statistics, Canberra

Summary

The (joint) cumulant of a set of (possibly coincident) random variables is defined as an alternating sum of moments with appropriate integral coefficients. By exploiting properties of the Möbius function of a partition lattice some basic results concerning cumulants are derived and illustrations of their use given.

1. Introduction

Cumulants were first defined and studied by the Danish scientist T. N. Thiele (1889, 1897, 1899) who called them *half-invariants* (halvinvarianter); see Hald (1981) for a review of this early work. The ready interpretability and descriptive power of the first few cumulants was evident to Thiele, as was their role in studying non-linear functions of random variables, and these aspects of their use have continued to be important to the present day, see Brillinger (1975, Section 2.3). In a sense which it is hard to make precise, all of the important aspects of (joint) distributions seem to be simpler functions of cumulants than of anything else, and they are also the natural tools with which transformations (linear or not) of systems of random variables (independent or not) can be studied when exact distribution theory is out of the question.

On the Combinatorics of Cumulants

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DEDICATED TO THE MEMORY OF GIAN-CARLO ROTA

We study cumulants by Umbral Calculus. Various formulae expressing cumulants by umbral functions are established. Links to invariant theory, symmetric functions, and binomial sequences are made.

PART 3:
RANDOM MEASURES
AND THE
ROTA-WALLSTROM THEORY

Approaching Rota-Wallstrom: Diagonals

- ⋆ Let $(Z, \mathcal{Z})$ be a Polish space, endowed with its Borel σ-field. Note that $\{x\} \in \mathcal{Z}$, for all $x \in Z$.
- ⋆ Fix $n \geq 2$. Given $\pi \in \mathcal{P}_n$, we set

$$\boxed{Z^n_\pi := \{(z_1, ..., z_n) \in Z^n : z_i = z_j \text{ if and only if } i \sim_\pi j\}}$$

where $i \sim_\pi j$ = "i and j are in the same block of π" (such a set is measurable).

- ⋆ Given $C \in \mathcal{Z}^n$, we set $C_\pi = C \cap Z^n_\pi$. Trivially,

$$C = \cup_{\pi \in \mathcal{P}_n} C_\pi \quad \text{(disjoint union).}$$

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Gaussian Random Measures

- ⋆ Let ν be a σ-finite non-atomic positive measure on $(Z, \mathcal{Z})$, and set $\mathcal{Z}_\nu := \{A \in \mathcal{Z} : \nu(A) < \infty\}$.
- ⋆ A **Gaussian measure** $G = \{G(A) : A \in \mathcal{Z}_\nu\}$ with **intensity** ν is a centered Gaussian family such that, for $A, B \in \mathcal{Z}_\nu$,

$$\mathbb{E}[G(A)G(B)] = \nu(A \cap B).$$

Recall:

$$\mathbb{P}(G(A) \in W) = \int_W \frac{e^{-x^2/2\nu(A)}}{\sqrt{2\pi\nu(A)}} dx.$$

- ⋆ Such an object exists !

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- ⋆ Write M to indicate either G or $\widehat{\eta}$.

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REMARKS

- ⋆ These are examples of **independently scattered** (or **completely random**) measures (see Kingman, 1967) – to which the whole theory basically applies.

- ⋆ We regard M as a **non-atomic Hilbert-space valued measure**, with values in a L^2 space. In particular, M is σ-additive: for A_i disjoint and $A = \cup_i A_i$,

$$M(A) = \sum_{i\geq 1} M(A_i),$$

  with convergence in $L^2(\mathbb{P})$; also, $M(\{x\}) = 0$, a.s.-$\mathbb{P}$ for every $x \in Z$.

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Stochastic Integrals of Order One

- ⋆ Integrating a function $f : Z \to \mathbb{R} \in L^2(\nu)$ is an easy task, solved by N. Wiener in the 30s.
  (1) First consider simple functions of the type $f(z) = \sum_{i=1}^{m} c_i \mathbf{1}_{A_i}(z)$, $A_i \in \mathcal{Z}_\nu$, and set
  $$I_1^M(f) := \int_Z f(z) M(dz) = \sum_{i=1}^{m} c_i \times M(A_i).$$
  (2) Observe that, for every f, g simple,
  $$E[I_1^M(f) \times I_1^M(g)] = \langle f, g \rangle_{L^2(\nu)}.$$
  (3) Extend the definition to every $f \in L^2(\nu)$ by a density argument. The L^2 closed vector space obtained in this way is the **first Wiener chaos** associated with M.
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- Things are more delicate when dealing with **multiple integrals**. Main difficulty: *M is a non-atomic vector-valued measure charging diagonals*.

M is Good!

Theorem (Engel, 1982)

*The measure M is "**good**": for every $n \geq 2$, there exists a unique collection of random variables*

$$M^n = \{M^n(C) : C \in \mathcal{Z}_\nu^n\} \subset L^2(\mathbb{P})$$

such that:

1. M^n *is σ-additive* [*in* $L^2(\mathbb{P})$] *;*
2. *for every cylinder* $C = A_1 \times \cdots \times A_n \in \mathcal{Z}_\nu^n$,

$$M^n(C) = M(A_1) \times \cdots \times M(A_n).$$

Stochastic and Diagonal Measures

- We can now define the following **stochastic measures**: for every $\pi \in \mathcal{P}_n$ and $C \in \mathcal{Z}_\nu^n$

$$M_\pi^n(C) := M^n(C_\pi), \quad M_{\succcurlyeq\pi}^n := \sum_{\sigma \succcurlyeq \pi} M_\sigma^n.$$

- For $n \geq 2$, the nth **diagonal measure** associated with M is given by: for every $A \in \mathcal{Z}$, $\Delta_1^M(A) = M(A)$, and

$$\Delta_n^M(A) = M_{\hat{1}}^n(A \times \cdots \times A).$$

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First Properties

- By Möbius inversion,

$$M^n_\pi := \sum_{\sigma \succcurlyeq \pi} \mu(\pi,\sigma) M^n_{\succcurlyeq \sigma}.$$

- This relation provides an implicit description of the measures M^n_π, via the relation:

$$\begin{aligned} M^n_{\succcurlyeq \pi}(A_1 \times \cdots \times A_n) &= \prod_{b \in \pi} M^{|b|}_{\hat{1}}(\mathbf{X}_{i \in b} A_i) \\ &= \prod_{b \in \pi} \Delta^M_{|b|}(\cap_{i \in b} A_i). \end{aligned}$$

Gauss and Poisson are Multiplicative

Theorem (Rota and Wallstrom, 1997)

For every $n \geq 2$ and every $\pi \in \mathcal{P}_n$, the product measure M^n is **multiplicative***: for every rectangle in $\mathcal{Z}_\nu^n$,*

$$\mathbb{E}[M_\pi^n(A_1 \times \cdots \times A_n)] = \prod_{b \in \pi} \mathbb{E}[\Delta_{|b|}^M(\cap_{j \in b} A_j)].)$$

Computation of Diagonal Measures

⋆ $\boxed{\Delta^2_G = \nu \text{ and } \Delta^n_G = 0 \text{ for every } n > 2.}$

(Idea of the proof: take a sequence of finite partitions $\{A_i^{(k)}\}_{k\geq 1}$ of A, such that $\max_i \nu(A_i^{(k)}) \to 0$. Then prove that

$$\sum_i G(A_i^{(k)})^2 \to \nu(A), \text{ and } \sum_i G(A_i^{(k)})^n \to 0, \forall n > 2.)$$

⋆ $\boxed{\Delta^n_{\widehat{\eta}} = \eta = \widehat{\eta} + \nu \text{ for every } n \geq 2.}$

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The Purely non-Diagonal Measure

Theorem (Rota and Wallstrom, 1997)

For every $n \geq 2$, the random measure $M_{\hat{0}}^n$ is the unique symmetric random measure with values in $L^2(\mathbb{P})$ such that

(I) $M_{\hat{0}}^n(C) = 0$, *for every* $C \subseteq Z_\pi^n$ *for some* $\pi \neq \hat{0}$.

(II) *For every set* $\tilde{C}$ *of the type*

$$\tilde{C} = \cup_{w \in \mathfrak{S}_n} C_{w(1)} \times \cdots \times C_{w(n)},$$

where the $C_i \in \mathcal{Z}_\nu$, $i = 1, ..., n$, *are pairwise disjoint,*

$$M_{\hat{0}}^n(\tilde{C}) = n! M(C_1) \times \cdots \times M(C_n).$$

Multiple Wiener-Itô integrals

One can indeed define **multiple (Wiener-Itô) integrals** with respect to $M_{\hat{0}}^n$, $n \geq 2$, for every symmetric $f \in L^2(\nu^n)$:

(1) First consider simple functions of the type $f(z_1, ..., z_n) = \sum_{i=1}^m a_i \mathbf{1}_{\tilde{C}_i}(z)$, where every $\tilde{C}_i \in \mathcal{Z}_\nu^n$ is a symmetrized rectangle with no diagonals as before, and set

$$I_n^M(f) := \int_{Z^n} f(z) M_{\hat{0}}^n(dz) = \sum_{i=1}^m a_i \times M_{\hat{0}}^n(\tilde{C}_i)$$

(2) Observe that, for every f, g simple and symmetric, $\mathbb{E}[I_n^M(f) \times I_m^M(g)] = n! \langle f, g \rangle_{L^2(\nu^n)} \mathbf{1}_{n=m}$.

(3) Extend the definition to every $f \in L_s^2(\nu^n)$ (symmetric and square-integrable) by a density argument. The $L^2(\mathbb{P})$ closed orthogonal vector spaces obtained in this way are the **Wiener chaoses** associated with M.

PART 4:
PRODUCT AND DIAGRAM FORMULAE

Product Formulae

Take integers $n_1, ..., n_k \geq 1$, write $n = n_1 + \cdots + n_k$, and consider the partition

$$\pi^* = \{[n_1], \{n_1 + 1, ..., n_1 + n_2\}, ..., \{n_1 + \cdots n_{k-1} + 1, ..., n\}\};$$

write $f_1 \otimes \cdots \otimes f_k$ for the **tensor product** of kernels $f_1, ..., f_k$.

Theorem (Product formulae; Rota & Wallstrom, 1997)
For adequate symmetric kernels $f_1, ..., f_k$ (for instance, simple and symmetric):

$$\prod_{j=1}^{k} I_{n_j}^M(f_j) = \sum_{\sigma \wedge \pi^* = \hat{0}} \int_{Z^n} (f_1 \otimes \cdots \otimes f_k) dM_\sigma^n.$$

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Proof #1 — Geometric

⋆ One has that

$$\prod_{j=1}^{k} I_{n_j}^{M}(f_j) = \int_{A^*} (f_1 \otimes \cdots \otimes f_k) M^n(dz_1, ..., dz_n)$$

where $A^* := \{(z_1, ..., z_n) : z_i \neq z_j, \ \forall i \neq j \text{ s.t. } i \sim_{\pi^*} j\}$.

⋆ To conclude:

$$A^* = \bigcup_{\sigma \wedge \pi^* = \hat{0}} Z_\sigma^n.$$

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Proof #2 — Algebraic

⋆ One has that

$$\prod_{j=1}^{k} I_{n_j}^{M}(f_j) = \prod_{j=1}^{k} \left\{ \sum_{\sigma_j \in \mathcal{P}([n_j])} \mu(\hat{0}, \sigma_j) M^{n}_{\succsim \sigma_j}(f_j) \right\}$$

$$= \sum_{\varrho \in [\hat{0}, \pi^\star]} \mu(\hat{0}, \varrho)\, M^{n}_{\succsim \varrho}(f_1 \otimes \cdots \otimes f_k)$$

$$= \sum_{\varrho \in [\hat{0}, \pi^\star]} \mu(\hat{0}, \varrho) \sum_{\gamma \succsim \varrho} M^{n}_{\gamma}(f_1 \otimes \cdots \otimes f_k).$$

⋆ Conclude by using the relation

$$\sum_{\varrho \in [\hat{0}, \pi^\star \wedge \gamma]} \mu(\hat{0}, \varrho) = \delta(\hat{0}, \pi^\star \wedge \gamma).$$

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$$\begin{aligned}\prod_{j=1}^{k} I_{n_j}^{M}(f_j) &= \prod_{j=1}^{k}\left\{\sum_{\sigma_j\in\mathcal{P}([n_j])}\mu(\hat{0},\sigma_j)M^{n}_{\succeq\sigma_j}(f_j)\right\}\\ &= \sum_{\varrho\in[\hat{0},\pi^\star]}\mu(\hat{0},\rho)\,M^{n}_{\succeq\varrho}(f_1\otimes\cdots\otimes f_k)\\ &= \sum_{\varrho\in[\hat{0},\pi^\star]}\mu(\hat{0},\rho)\,\sum_{\gamma\succeq\varrho}M^{n}_{\gamma}(f_1\otimes\cdots\otimes f_k).\end{aligned}$$

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PROOF #2 — ALGEBRAIC

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$$\sum_{\varrho\in[\hat{0},\pi^\star\wedge\gamma]}\mu(\hat{0},\rho)=\delta(\hat{0},\pi^\star\wedge\gamma).$$

□

Contractions

In addition to graphical representations, multiplication formulae can be neatly represented in terms of **contraction operators**: for every symmetric $f \in L_s^2(\nu^p)$, $g \in L^2(\nu^q)$ and every $r = 0, ..., \min(p, q)$

$$f \otimes_r g(x_1, ..., x_{p+q-2r}) :=$$
$$\int_{Z^r} f(\mathbf{a}_r, x_1, ..., x_{p-r}) g(\mathbf{a}_r, x_{q-r+1}, ..., x_{p+q-2r}) \nu^r(d\mathbf{a}_r).$$

CONTRACTIONS

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Product Formulae: Gaussian

Let $p, q \geq 1$ and let $f \in L^2(\nu^p),\ g \in L^2(\nu^q)$ be symmetric.

Theorem (Product of Gaussian Integrals)

$$I_p^G(f) \times I_q^G(g) = \sum_{r=0}^{\min(p,q)} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(\widetilde{f \otimes_r g}).$$

For $n = p + q$ and $\pi^* = \{\{1, ..., p\}, \{p+1, ..., q\}\}$, the coefficient $r!\binom{p}{r}\binom{q}{r}$ counts those $\sigma \in \mathcal{P}_n$ such that $\sigma \wedge \pi^* = \hat{0}$ and σ has r blocks with two edges and $p + q - 2r$ singletons.

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Product Formulae: Poisson

- $\star$ In the Poisson case, multiplication formulae have a more complex form, due to the fact that $\Delta_n^{\widehat{\eta}} = \widehat{\eta} + \nu$, $n \geq 2$.
- $\star$ **Example:** For symmetric f, g in two variables:

$$I_2^{\widehat{\eta}}(f) \times I_2^{\widehat{\eta}}(g) = I_4^{\widehat{\eta}}(f \otimes g) + I_3^{\widehat{\eta}}(f \star_1^0 g) + I_2^{\widehat{\eta}}(f \otimes_1 g)$$
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where (Y. Kabanov's notation, 1976)

$$f \star_1^0 g(x,y,z) = f(x,y)g(x,z), \quad f \star_2^0 g(x,y) = f(x,y)g(x,y),$$

and $f \star_2^1 g(z) = \int_Z f(x,z)g(x,z)\nu(dx)$.

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Orthogonal Polynomials: Gaussian Case

⋆ Recall that the **Hermite polynomials** $\{H_n(x) : n \geq 0\}$ on the real line are given by the recursive relation $H_0 = 1$ and

$$H_n(x) = \delta H_{n-1}(x), \quad \delta f(x) = xf(x) - f'(x).$$

⋆ For instance, $H_1(x) = x$, $H_2(x) = x^2 - 1$, ... We easily verify that

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- ⋆ Now select $h \in L^2(\mu)$ such that $\|h\|_{L^2(\mu} = 1$. For $n \geq 1$ we apply the multiplication formula to the two integrals $I_n^G(h^{\otimes n})$ and $I_1^G(h)$:

$$I_n^G(h^{\otimes n}) \times I_1^G(h) = I_{n+1}^G(h^{\otimes(n+1)}) + nI_{n-1}^G(h^{\otimes(n-1)}).$$

- ⋆ Since $I_1^G(h) = H_1[I_1^G(h)]$, we obtain the important formula

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- ⋆ **Remark:** In the Poisson case, similar connections (but less elegant and less exhaustive) can be established with the class of **Charlier polynomials**.

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Chaotic Decomposition

The following statement applies to $M = G, \widehat{\eta}$.

Theorem (Chaotic decomposition)

For every $F \in L^2(\sigma(M), \mathbb{P})$,

$$F = E[F] + \sum_{n \geq 1} I_n^M(f_n),$$

where $f_n \in L_s^2(\nu^n)$, the decomposition is unique, and the series is converging in $L^2(\mathbb{P})$. In compact Fock space notation,

$$L^2(\sigma(M), \mathbb{P}) = \mathbb{R} \oplus \bigoplus_{n \geq 1} C_n \simeq \mathbb{R} \oplus \bigoplus_{n \geq 1} \sqrt{n!} L_s^2(\nu^n),$$

where C_n stands for the nth Wiener chaos of M.

Diagram Formulae: Moments

We consider, integers $n_1, ..., n_k \geq 1$, write $n = n_1 + \cdots + n_k$, and consider the partition $\pi^* \in \mathcal{P}_n$ given by

$$\pi^* = \{\{1, ..., n_1\}, \{n_1 + 1, ..., n_1 + n_2\}, ..., \{n_1 + \cdots n_{k-1} + 1, ..., n\}\}.$$

Theorem (Diagram Formulae for Moments)

For adequate symmetric kernels $f_1, ..., f_k$ (for instance, simple and symmetric):

$$\mathbb{E}\left[\prod_{j=1}^{k} I_{n_j}^M(f_j)\right] = \sum_{\sigma \wedge \pi^* = \hat{0}} \int_{Z^n} (f_1 \otimes \cdots \otimes f_k) \left[\bigotimes_{b \in \sigma} d \left\langle \Delta_{|b|}^M \right\rangle\right],$$

where: $\left\langle \Delta_{|b|}^M \right\rangle = \mathbb{E}[\Delta_{|b|}^M]$.

Diagram Formulae: Cumulants

We consider, integers $n_1, ..., n_k \geq 1$, write $n = n_1 + \cdots + n_k$, and consider the partition $\pi^* \in \mathcal{P}_n$ given by

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Theorem (Diagram Formulae for Cumulants)

For adequate symmetric kernels $f_1, ..., f_k$ (for instance, simple and symmetric):

$$\chi(I^M_{n_1}(f_1), ..., I^M_{n_k}(f_k)) = \sum_{\sigma \wedge \pi^* = \hat{0}\,;\, \sigma \vee \pi^* = \hat{1}} \int_{Z^n} (f_1 \otimes \cdots \otimes f_k) \left[\bigotimes_{b \in \sigma} d \left\langle \Delta^M_{|b|} \right\rangle \right].$$

The difference with the expression for moments resides in the additional constraint $\sigma \vee \pi^* = \hat{1}$.

PROOFS (SKETCH).

- ⋆ The formula for moments is a direct consequence of multiplicativity.
- ⋆ The formula for cumulants follows from the relation below (which is a – not so immediate! – consequence of Leonov and Shyraev formulae):

$$\chi\left(I^M_{n_1}(f_1), \cdots, I^M_{n_k}(f_k)\right) = \sum_{\pi^* \preceq \rho = (r_1,\ldots,r_l) \in \mathcal{P}_n} \mu(\rho, \hat{1}) \sum_{\substack{\gamma \preceq \rho \\ \gamma \wedge \pi^* = \hat{0}}} \bigotimes_{b \in \gamma} \left\langle \Delta^M_{|b|} \right\rangle (f_1 \otimes \cdots \otimes f_k),$$

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Proofs (Sketch).

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Using the Formula: Gaussian Case

- ⋆ Recall that $\Delta_2^G = \nu$ and $\Delta_n^G = 0$, for $n > 2$, meaning that **the only contribution to the formulae comes from non-flat perfect matchings**.
- ⋆ Consider kernels $f_1, ..., f_k$ and build the tensor product $f_1 \otimes \cdots \otimes f_k$. This a function of $n = n_1 + \cdots + n_k = n$ variables; we assume n even (*otherwise the formula gives zero*).
- ⋆ For every perfect matching $\gamma \in \mathcal{M}_n$, build the function F_γ, of $n/2$ variables by **identifying two variables** x_i, x_j **in the argument of** $f_1 \otimes \cdots \otimes f_k$ **whenever** $i \sim_\gamma j$.
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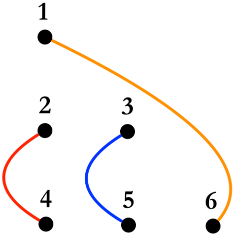
Example



Consider the case $n = 6$, and

$$\pi^\star = \{\{1\},\{2,3\},\{4,5,6\}\}.$$

Then, the perfect matching

$$\gamma = \{\{1,6\},\{2,4\},\{3,5\}\}$$

verifies $\gamma \wedge \pi^\star = \hat{0}$, and $\gamma \vee \pi^\star = \hat{1}$.

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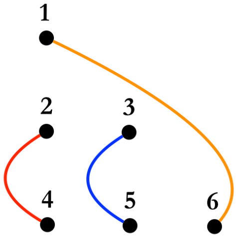

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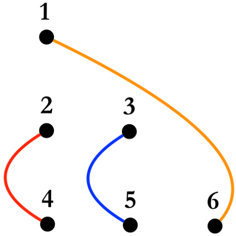

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Double Gaussian Integrals, I

⋆ For every $m \geq 2$, we want to compute

$$\chi_m(I_2^G(f)) = \chi(\underbrace{I_2^G(f), ..., I_2^G(f)}_{m \text{ times}}),$$

for a generic symmetric kernel f.

⋆ By the theorem,

$$\chi_m(I_2^G(f)) = \sum_{\substack{\gamma \in \mathcal{M}_{2m} \\ \gamma \wedge \pi^* = \hat{0}\,;\, \gamma \vee \pi^* = \hat{1}}} \int_{Z^m} F_\gamma \, d\nu^m,$$

where $\pi^\star = \{\{1,2\},\{3,4\},...,\{2m-1,2m\}\} = \{\{2j-1,2j\} : j = 1,...,m\}$, and F_γ is obtained from

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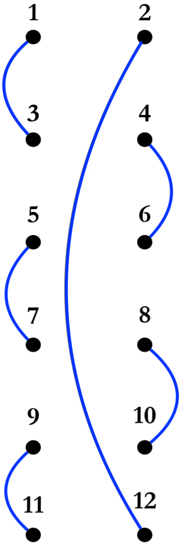

For every $m \geq 2$, $\mathcal{M}_{2m}$ contains exactly $2^{m-1}(m-1)!$ solutions γ to the system

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Up to a permutation of variables, each solution generates the same function F_γ.

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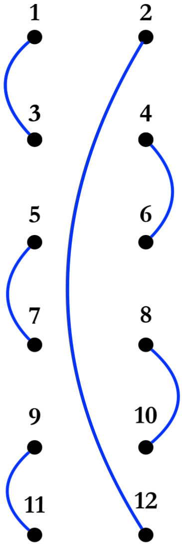

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Double Gaussian Integrals, III

Proposition

For f symmetric and square-integrable,

$$\chi_m(I_2^G(f)) = 2^{m-1}(m-1)! \times \int_{Z^m} f(x_1, x_2) f(x_2, x_3) \cdots f(x_m, x_1) \nu^m(dx_1, \cdots, dx_m) = 2^{m-1}(m-1)! \mathbf{Trace}(\mathbb{H}_f^m),$$

where $\mathbb{H}_f^m$ is the Hilbert-Schmidt operator

$$h \mapsto \mathbb{H}_f^m(h) := \int_Z h(x) f(\bullet, x) \nu(dx).$$

USING THE FORMULA: POISSON CASE

- ⋆ Recall that $\langle \Delta_n^{\widehat{\eta}} \rangle = \nu$ for $n \geq 2$, meaning that **non-flat partitions with no singletons contribute to the formulae**. Such a class is denoted by $\mathcal{P}_n^0$.
- ⋆ Consider kernels $f_1, ..., f_k$ and build the tensor product $f_1 \otimes \cdots \otimes f_k$. This a function of $n = n_1 + \cdots + n_k = n$ variables.
- ⋆ For every $\gamma \in \mathcal{P}_n^0$, build the function F_γ **by identifying those variables in the argument of $f_1 \otimes \cdots \otimes f_k$ that are in the same block of γ.**
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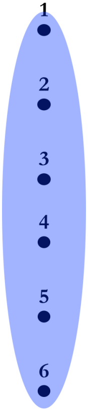

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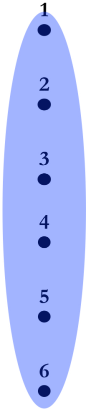

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PART 5:
LIMIT THEOREMS

A Classical Result

Theorem (Method of moments and cumulants)
Let X be a random variable whose distribution is determined by the moments, and let $\{X_n : n \geq 1\}$ be a sequence of random variables with finite moments of all orders such that, for all integers $m \geq 1$, either

$$\mathbb{E}[X_n^m] \longrightarrow \mathbb{E}[X^m], \quad n \to \infty,$$

or

$$\chi_m(X_n) \longrightarrow \chi_m(X), \quad n \to \infty.$$

Then,

$$X_n \xrightarrow{\textbf{LAW}} X.$$

Remark. If $X \sim \mathscr{N}(0,1)$, then $\chi_2(X) = \mathbf{Var}(X) = 1$ and $\chi_m(X) = 0$ for every $m \neq 2$.

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Fourth Moment Theorem — Gaussian Case

Theorem (Nualart and Peccati, 2005)
Let $q \geq 2$ and $\{I_q^G(f_n) : n \geq 1\}$ be such that $\mathbb{E}[I_q^G(f_n)^2] = 1$. Then, the following are equivalent, as $n \to \infty$:

(1) $I_q^G(f_n) \xrightarrow{\text{LAW}} X \sim \mathscr{N}(0,1)$

(2) $\chi_4(I_q^G(f_n)) \to 0$

(3) $\mathbb{E}(I_q^G(f_n)^4) \to 3$

Remarks. The implication **(1)** $\Rightarrow$ **(2)** $\Leftrightarrow$ **(3)** are trivial.
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Fourth Moment Theorem — Poisson Case

Theorem (Döbler and Peccati, 2017; Döbler, Vidotto and Zheng, 2017)
Let $q \geq 1$ *and* $\{I_q^{\widehat{\eta}}(f_n) : n \geq 1\}$ *be such that* $\mathbb{E}[I_q^{\widehat{\eta}}(f_n)^2] = 1$. *If* $\chi_4(I_q^{\widehat{\eta}}(f_n)) \to 0$, *then*

$$I_q^{\widehat{\eta}}(f_n) \xrightarrow{\textbf{LAW}} X \sim \mathscr{N}(0,1).$$

Remark. Free version by Bourguin and Peccati (2014).

- ⋆ Call $\mathcal{A}$ the algebra composed of finite linear combinations of multiple integrals $I_p^G(g)$, where $p \geq 0$ and g is symmetric.

- ⋆ Write C_m for the mth Wiener chaos of G ($C_m \subset \mathcal{A}$).

- ⋆ The product formula yields that

$$I_q^G(f_n)^2 = \sum_{m=0}^{q} \mathbf{proj}(I_q(f_n)^2 \mid C_{2m}),$$

with $\mathbf{proj}(I_q(f_n)^2 \mid C_0) = \mathbb{E}[I_q^G(f_n)^2] = 1$ by definition.

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Proof — Fourth Moments

⋆ By orthogonality,

$$\mathbb{E}[I_q^G(f_n)^4] = 1 + \sum_{m=1}^{q-1} \mathbb{E}[\mathbf{proj}(I_q(f_n)^2 \mid C_{2m})^2] + \mathbb{E}[\mathbf{proj}(I_q(f_n)^2 \mid C_{2q})^2]$$

Lemma (Nualart & Peccati, 2005)

$$\mathbb{E}[\mathbf{proj}(I_q(f_n)^2 \mid C_{2q})^2] = 2 + (q!)^2 \sum_{r=1}^{q-1} \binom{q}{r}^2 \|f_n \otimes_r f_n\|;^2$$

$$\chi_4(I_q(f_n)) \geq K_q \max_{r=1,\dots,q-1} \|f_n \otimes_r f_n\|^2;$$

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Proof — Carré-du-Champ

- ⋆ Introduce the operator L (**generator of the Ornstein-Uhlenbeck semigroup**) on $\mathcal{A}$ by the relation

$$L(I_p^G(g)) = -pI_p^G(g), \quad \text{and then extend by linearity.}$$

- ⋆ Define the **carré-du-champ operator**

$$\Gamma(F, J) = \frac{1}{2}\left[L(FJ) - JLF - FLJ\right],$$

for which

$$\mathbb{E}[FLJ] = -\mathbb{E}[\Gamma(F, J)].$$

- ⋆ **Fact.** Γ is diffusive: for every polynomial P and every $F, J \in \mathcal{A}$

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- ⋆ Write $F = I_q^G(f_n)$, and choose a polynomial P. Assume that F and $X \sim \mathscr{N}(0,1)$ are independent.
- ⋆ Then, $|\mathbb{E}[P(F)] - \mathbb{E}[P(X)]| \leq \|\Psi'\|_\infty$, where ($t \in [0,1]$)

$$\Psi(t) := \mathbb{E}[P(\sqrt{t}F + \sqrt{1-t}X)] := \mathbb{E}[P(F_t)].$$

- ⋆ One has that

$$\begin{aligned} |\Psi'(t)| &= \frac{1}{2}\left|\frac{1}{q}\mathbb{E}[\Gamma(F,F)P''(F_t)] - \mathbb{E}[P''(F_t)]\right| \\ &\leq K\mathbf{Var}[\Gamma(F,F)/q]^{1/2}. \end{aligned}$$

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$$\frac{\Gamma(F,F)}{q} = \frac{1}{2q}\{L(F^2) + 2qF^2\} = \sum_{m=0}^{q-1} \frac{q-m}{q}\mathbf{proj}(F^2 \mid C_{2m}).$$

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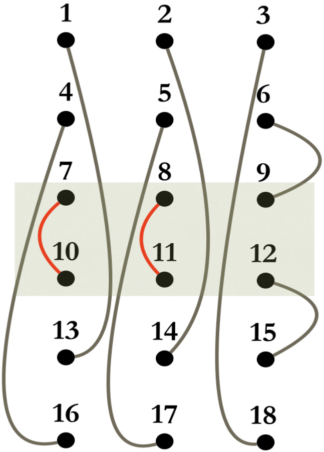

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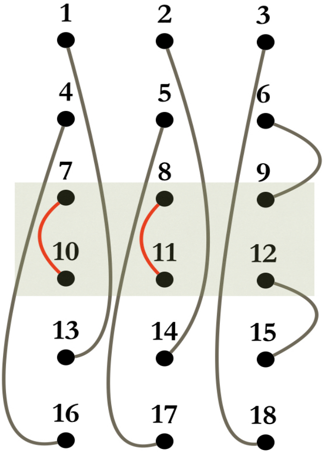

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Joint Gaussianity — Peccati & Tudor, 2005

Theorem (Joint Gaussian Convergence)

For $d \geq 2$, let

$$\mathbf{F}_n = (F_{n,1}, ..., F_{n,d}) := (I^G_{q_1}(f^1_n), ..., I^G_{q_d}(f^d_n)), \quad n \geq 1,$$

be such that $\mathbf{Cov}(\mathbf{F}_n) \to \Sigma \geq 0$, as $n \to \infty$. Then, the following are equivalent, as $n \to \infty$,

(1) $\mathbf{F}_n \overset{\mathbf{LAW}}{\to} \mathbf{X} \sim \mathscr{N}_d(0, \Sigma)$;

(2) *For every $i = 1, ...d$, $F_{n,i} \overset{\mathbf{LAW}}{\to} X_i \sim \mathscr{N}(0, \Sigma(i,i))$.*

Remark. Similar results on the Poisson space: Peccati and Zheng (2010), Döbler, Vidotto and Zheng (2017).
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Independence – Nourdin and Rosinski, 2010 ; Nourdin, Nualart and Peccati, 2013

Theorem (Asymptotic independence)

Let

$$F_n = I_q^G(f_n), \quad J_n = I_p^G(g_n), \quad n \geq 1.$$

be such that F_n and J_n converge in distribution. Then, the following are equivalent as $n \to \infty$:

(1) *F_n and J_n are asymptotically independent ;*

(2) $\mathbf{Cov}(F_n^2, J_n^2) \to 0.$

Remark.

Results of this type are still unkown on the Poisson space.
Free version: Bourguin and Nourdin (2017)

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(1) *F_n and J_n are asymptotically independent ;*

(2) $\mathbf{Cov}(F_n^2, J_n^2) \to 0.$

Remark.

Results of this type are still unkown on the Poisson space.
Free version: Bourguin and Nourdin (2017)

PART 6:
A GEOMETRIC EXAMPLE

A Model (Berry, 1977)

- ⋆ Fix $E > 0$. The **Berry random wave model** on $\mathbb{R}^2$ with parameter E, written

$$B_E = \{B_E(x) : x \in \mathbb{R}^2\},$$

is defined as the unique (in law) centred, isotropic Gaussian field on $\mathbb{R}^2$ such that

$$\Delta B_E + 4\pi^2 E \cdot B_E = 0, \text{ where } \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.$$

- ⋆ Equivalently, $\mathbb{E}[B_E(x)B_E(y)] = J_0(2\pi\sqrt{E}\|x-y\|)$ (J_0 = **Bessel function of the 1st kind**) or

$$B_E(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{S}^1} e^{2i\pi\sqrt{E}\langle z,x\rangle} G(dz),$$

where $G :=$ **Hermitian Gaussian measure** on the unit circle.

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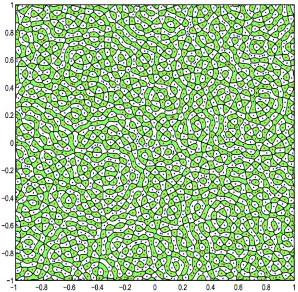
NODAL SETS

We are interested in the high-energy (as $E \to \infty$) geometry of the **nodal sets** (components are the **nodal lines**):

$$B_E^{-1}(\{0\}) \cap \mathcal{D} := \{x \in \mathcal{D} : B_E(x) = 0\},$$

where $\mathcal{D}$ is a compact set with piecewise smooth boundary. In particular, in

$$L_E := \text{length}\, B_E^{-1}(\{0\}) \cap \mathcal{D}$$



SOME MOTIVATIONS

- ⋆ Geometric study of **excursion sets** of isotropic random fields.
- ⋆ An amplification of **Berry's universality conjecture** (1977) states that the **high-energy** behaviour of Laplace eigenfunctions on a Riemaniann surface coïncides with the average behaviour of the Random Wave Model on a comparable planar domain (see Zelditch, 2009). Used to heuristically test open problems on the geometry of deterministic nodal sets, like e.g. **Yau's conjecture**.

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Mean and Variance

- *Berry* (J. Stat. Physics A, 2002) : semi-rigorous computations lead to:

$$\mathbb{E}[L_E] = \frac{2\pi\sqrt{E}}{2\sqrt{2}}, \quad \mathbf{Var}(L_E) \sim \frac{\text{area}\,\mathcal{D}}{512\pi} \log E,$$

  although the natural guess for the order of the variance is $\sim \sqrt{E}$. Such a variance reduction "*... results from a cancellation whose meaning is still obscure...* " (Berry (2002), p. 3032).

- Constants rigorously confirmed in the model of **random spherical harmonics** (Wigman (Comm. Math. Ph., 2007)). Similar cancellations proved on the flat torus by Krishnapur, Kurlberg and Wigman (Ann. Math., 2013).

- Write

$$\widetilde{L}_E := \frac{L_E - \mathbb{E}(L_E)}{\mathbf{Var}(L_E)^{1/2}}.$$

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A Result

Theorem (Nourdin, Peccati & Rossi, 2017)

1. **(Cancellation)** *For every fixed* $E > 0$*,*

$$\mathrm{proj}(L_E \,|\, \mathrm{Chaos}_{2q+1}) = 0, \quad q \geq 0,$$

   and $\mathrm{proj}(\widetilde{L}_E \,|\, \mathrm{Chaos}_2)$ *reduces to a "negligible boundary term", as* $E \to \infty$*.*

2. **(4^{th} chaos dominates)** *Let* $E \to \infty$*. Then,*

$$\widetilde{L}_E = \mathrm{proj}(\widetilde{L}_E \,|\, \mathrm{Chaos}_4) + o_{\mathbb{P}}(1).$$

3. **(CLT)** *As* $E \to \infty$*,*

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Plot Twist

- ⋆ The fourth chaos dominance **continues to hold on more general manifolds** (spheres, tori, ...), and generates variance cancellations.
- ⋆ The nodal length on the sphere **verifies a CLT** (Marinucci, Rossi and Wigman, 2017)
- ⋆ The nodal length on the flat torus **verifies a non-universal & non-central limit theorem** (Marinucci, Peccati, Rossi and Wigman, GAFA 2017)
- ⋆ **Locally**, everything behaves like Berry's model (and therefore **has Gaussian fluctuations**), see Canzani & Hanin, 2016.
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Final Words

Thank you for your attention !

(Sorry I have to leave...)