An application to random geometric graphs
(Santander, July 17-19 2017, by G. Peccati)

I – Setting. Fix an integer $d \geq 2$. For every $n \geq 1$, we denote by η_n a Poisson measure on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ with intensity given by

$$\nu_n(dx) = n \times p(x)dx,$$

where p is a continuous and bounded probability density on $\mathbb{R}^d$; as in the lectures, we set $\widehat{\eta}_n = \eta_n - \nu_n$. We consider a sequence of positive numbers $\{s_n : n \geq 1\}$; for every n, we define the (undirected) **random geometric graph** $G_n := (V_n, E_n)$ as follows :

$$V_n = \text{supp}(\eta_n),$$

where $\text{supp}(\eta_n)$ indicates the support of η_n (in our case, this is the random set composed of those $x \in \mathbb{R}^d$ that are charged by η_n), and $\{x, y\} \in E_n$ if and only if $\|x - y\| \in (0, s_n]$ (note that, by construction, this graph has no loops). The graph G_n is sometimes called a (Euclidean) **Gilbert graph**.

Remark on notation. All norms below have to be understood with respect to the intensity measure ν_n or its powers (the context will be always clear).

Our aim in this document is to understand the behaviour of the quantity

$$\mathbf{E}_n := |E_n| \quad \text{(number of edges in } G_n\text{)},$$

as $n \to \infty$, under the particular assumption that

$$n s_n^d \to 0 \quad \text{(implying } s_n \to 0\text{)}. \tag{0.1}$$

II – Chaos expansion and mean. For every n, write

$$f_n(x, y) = \frac{1}{2}\mathbf{1}_{\{\|x-y\| \in (0, s_n]\}}.$$

Our first step is the simple observation that

$$\begin{aligned}
\mathbf{E}_n &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f_n(x, y)\eta_n(dx)\eta_n(dy) \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f_n(x, y)\nu_n(dx)\nu_n(dy) + I_1^{\widehat{\eta}_n}(g_n) + I_2^{\widehat{\eta}_n}(f_n) \\
&= \nu_n^2(f_n) + I_1^{\widehat{\eta}_n}(g_n) + I_2^{\widehat{\eta}_n}(f_n),
\end{aligned}$$

where $g_n(x) = 2\int_{\mathbb{R}^d} f_n(x,y)\nu_n(dy)$, from which it follows that

$$\begin{aligned}\mathbb{E}[\mathbf{E}_n] &= \nu^n(f_n) = \frac{n^2}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p(x)p(y)\mathbf{1}_{\{0<\|x-y\|\le s_n\}}dxdy \\ &= \frac{n^2 s_n^d}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p(a)p(a+s_n b)\mathbf{1}_{\{0<\|b\|\le 1\}}dadb \\ &\sim \alpha\times n^2 s_n^d\end{aligned}$$

where we used the change of variables $a = x$ and $b = (x-y)/s_n$, and

$$\alpha := \frac{\mathrm{Vol}(B_1)}{2}\int_{\mathbb{R}^d} p(a)^2 da,$$

with B_1 the unit ball in $\mathbb{R}^d$.

III – Case $n^2 s_n^d \to \infty$. Computations similar to those above show that

$$\begin{aligned}\mathbb{E}[I_1^{\widehat{\eta}_n}(g_n)^2] = \|g_n\|^2 &= 4n^3\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p(x)p(y)p(z)\mathbf{1}_{\{\|x-y\|\le s_n\}}\mathbf{1}_{\{\|x-z\|\le s_n\}}dxdydz \\ &\sim \beta n^3(s_n^d)^2 = o(n^2 s_n^d), \qquad (0.2)\end{aligned}$$

where $\beta > 0$ is some finite absolute constant whose value is immaterial for our discussion. On the other hand

$$\mathbb{E}[I_2^{\widehat{\eta}_n}(f_n)^2] = 2\|f_n\|^2 = \nu_n(f_n) \sim \alpha\, n^2 s_n^d.$$

This computation yields in particular that

$$\mathbf{Var}(\mathbf{E}_n) \sim \mathbb{E}[\mathbf{E}_n] \sim \alpha n^2 s_n^d,$$

and consequently that the limit in distribution of the normalized quantity

$$\widetilde{\mathbf{E}}_n := \frac{\mathbf{E}_n - \mathbb{E}[\mathbf{E}_n]}{\mathbf{Var}(\mathbf{E}_n)^{1/2}}$$

is the same as that of

$$\frac{I_2^{\widehat{\eta}_n}(f_n)}{(\alpha\, n^2 s_n^d)^{1/2}} = I_2^{\widehat{\eta}_n}(h_n) := Y_n,$$

where $h_n := f_n/(\alpha\, n^2 s_n^d)^{1/2}$.

Our aim is now to prove that, as $n\to\infty$, Y_n converges in law to a standard Gaussian random variable. In order to prove this, we write

$$\pi^\star = \{\{1,2\},\{3,4\},\{5,6\},\{7,8\}\} \in \mathcal{P}_8,$$

and recall from the lectures that the desired conclusion would follow, if we could prove that, for every $\gamma \in \mathcal{P}_8$ such that $\gamma \vee \pi^\star = \hat{1}$ and $\gamma \wedge \pi^\star = \hat{0}$,

$$\int_{(\mathbb{R}^d)^{|\gamma|}} (H_n)_\gamma \, d\nu_n^{|\gamma|} \longrightarrow 0, \quad n \to \infty,$$

where $(H_n)_\gamma$ is the function in $|\gamma|$ variables, obtained by identifying variables in the same block of γ into the argument of the tensor product $h_n \otimes h_n \otimes h_n \otimes h_n$ (which is a function in 8 variables). We will say that two partitions $\sigma, \rho \in \mathcal{P}_n$ have the same type if $|\sigma| = |\rho|$, and

$$(H_n)_\sigma(x_1, ..., x_{|\sigma|}) = (H_n)_\rho(x_{w(1)}, ..., x_{w(|\sigma|)}),$$

for some permutation w of $\{1, ..., |\sigma|\}$. It is easily checked that, if $\gamma \in \mathcal{P}_n$ verifies $\gamma \vee \pi^\star = \hat{1}$ and $\gamma \wedge \pi^\star = \hat{0}$, then necessarily γ has the same type as one of the partitions $\gamma_1, \gamma_2, \gamma_3$ and γ_4 whose blocks are represented as gray shapes in Fig. 1.

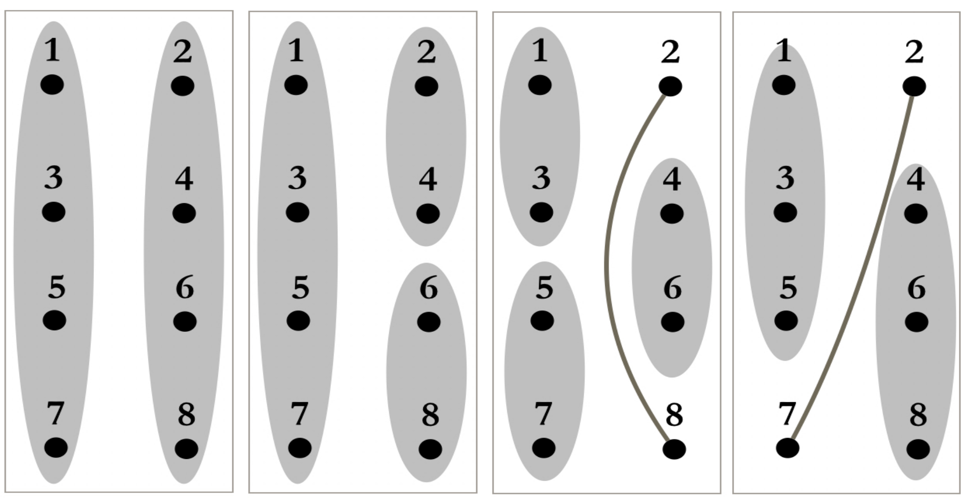

FIGURE 1 – The partitions $\gamma_1, \gamma_2, \gamma_3$ and γ_4

If $\gamma \in \mathcal{P}_n$ verifies $\gamma \vee \pi^\star = \hat{1}$ and $\gamma \wedge \pi^\star = \hat{0}$, then $|\gamma| \in \{2, 3, 4\}$, and computations similar to the ones above yield that

$$\int_{(\mathbb{R}^d)^{|\gamma|}} (H_n)_\gamma \, d\nu_n^{|\gamma|} \sim \kappa \frac{n^{|\gamma|}(s_n^d)^{|\gamma|-1}}{n^4 (s_n^d)^2} \to 0, \quad n \to \infty,$$

where κ is some absolute positive constant.

As anticipated, this relation allows us to conclude that, as $n \to \infty$, both Y_n and $\widetilde{\mathbf{E}}_n$ converge in distribution to a standard Gaussian random variable.

IV – Case $n^2 s_n^d \to c \in [0,\infty)$. If $c = 0$, then it is easily seen that, as $n \to \infty$, the random variable $\mathbf{E}_n$ converges to 0 in $L^1(\mathbb{P})$. We can therefore assume that $n^2 s_n^d \to c \in (0,\infty)$. In view of the computations in the previous sections, we know already that

$$\mathbf{Var}(\mathbf{E}_n), \quad \mathbb{E}[\mathbf{E}_n] \longrightarrow \alpha c,$$

and also that (since (0.2) continues to hold when $n^2 s_n^d$ converges to a finite limit) the limit in distribution of $\mathbf{E}_n$ coincides with that of

$$Z_n := \alpha c + I_2^{\widehat{\eta}_n}(f_n), \quad n \geq 1.$$

We will prove that Z_n converges in distribution to a Poisson random variable with parameter αc. In order to see this, we fix an integer $m \geq 2$, and consider the partition

$$\pi^\star = \{\{1,2\},\{3,4\},...,\{2m-1,2m\}\} \in \mathcal{P}_{2m}.$$

According to the lectures,

$$\chi_m(Z_n) \;=\; \sum \int_{(\mathbb{R}^d)^{|\gamma|}} (F_n)_\gamma \, d\nu_n^{|\gamma|},$$

where the sum runs over all $\gamma \in \mathcal{P}_{2m}$ such that $\gamma \wedge \pi^\star = \hat{0}$, and $\gamma \vee \pi^\star = \hat{1}$, and $(F_n)_\gamma$ is the function in $|\gamma|$ variables obtained by identifying those elements in the argument of

$$\underbrace{f_n \otimes \cdots \otimes f_n}_{m \text{ times}}$$

that are in the same block of γ. Computations analogous to those performed above show that, if γ verifies the desired relations and $|\gamma| > 2$, then

$$\int_{(\mathbb{R}^d)^{|\gamma|}} (F_n)_\gamma \, d\nu_n^{|\gamma|} \to 0.$$

On the other hand, there are exactly 2^{m-1} partitions with two blocks verifying $\gamma \wedge \pi^\star = \hat{0}$, and $\gamma \vee \pi^\star = \hat{1}$, and for each of them

$$\int_{(\mathbb{R}^d)^{|\gamma|}} (F_n)_\gamma \, d\nu_n^{|\gamma|} \to \frac{\mathrm{Vol}(B_1)}{2^m} \int_{\mathbb{R}^d} p(a)^2 da.$$

This implies that $\chi_m(Z_n) \to \alpha c$ for every integer $m \geq 1$, and therefore that Z_n converges in distribution to a Poisson random variable with parameter αc (recall that the Poisson distribution is determined by its moments).

V – Further remarks. The first paper using the asymptotic properties of multiple Wiener-Itô integrals for dealing with models of stochastic geometry is :

$\star$ M. Reitzner and M. Schulte (2013). Central limit theorems for U-statistics of Poisson point processes, *Ann. Probab.* **41**, 3879-3909

The literature on the matter has evolved very quickly, see e.g. the recent collective monograph

$\star$ G. Peccati and M. Reitzner (Editors) (2016). *Stochastic analysis for Poisson point processes : Malliavin calculus, Wiener-Itô chaos expansions and stochastic geometry*. Springer-Verlag.

A parallel study of convergence results, similar to the ones described in this section, both for classical and non-commutative Poisson measures can be found here :

$\star$ S. Bourguin and G. Peccati (2014). Semicircular limits on the free Poisson chaos : counterexamples to a transfer principle. *Journal of Functional Analysis*, **267**(4), 963-997