Combinatorial Aspects of Free Probability and Free Stochastic Calculus

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"Non-Commutative Distributions in Free Probability"

Part I:

Free Probability and Non-Crossing Partitions

Some History

1985 Voiculescu introduces ”freeness” in the context of isomorphism problem of free group factors

1991 Voiculescu discovers relation with random matrices (which leads, among others, to deep results on free group factors)

1994 Speicher develops combinatorial theory of freeness, based on ”free cumulants”

later ... many new results on operator algebras, eigenvalue distribution of random matrices, and much more ....

Definition of Freeness

Let $(\mathcal{A}, \varphi)$ be **non-commutative probability space**, i.e., $\mathcal{A}$ is a unital algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ is unital linear functional (i.e., $\varphi(1) = 1$)

Unital subalgebras $\mathcal{A}_i$ $(i \in I)$ are **free** or **freely independent**, if $\varphi(a_1 \cdots a_n) = 0$ whenever

- $a_i \in \mathcal{A}_{j(i)}, \qquad j(i) \in I \quad \forall i, \qquad j(1) \neq j(2) \neq \cdots \neq j(n)$

- $\varphi(a_i) = 0 \quad \forall i$

Random variables $x_1, \ldots, x_n \in \mathcal{A}$ are free, if their generated unital subalgebras $\mathcal{A}_i :=$ algebra$(1, x_i)$ are so.

What is Freeness?

Freeness between A and B is an infinite set of equations relating various moments in A and B:

$$\varphi\Big(p_1(A)q_1(B)p_2(A)q_2(B)\cdots\Big) = 0$$

Basic observation: freeness between A and B is actually a **rule for calculating mixed moments** in A and B from the moments of A and the moments of B:

$$\varphi\Big(A^{n_1}B^{m_1}A^{n_2}B^{m_2}\cdots\Big) = \text{polynomial}\Big(\varphi(A^i), \varphi(B^j)\Big)$$

Example:

$$\varphi\Big(\big(A^n - \varphi(A^n)1\big)\big(B^m - \varphi(B^m)1\big)\Big) = 0,$$

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thus

$$\varphi(A^nB^m) - \varphi(A^n\cdot 1)\varphi(B^m) - \varphi(A^n)\varphi(1\cdot B^m) + \varphi(A^n)\varphi(B^m)\varphi(1\cdot 1) = 0,$$

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$$\varphi(A^nB^m)-\varphi(A^n\cdot 1)\varphi(B^m)-\varphi(A^n)\varphi(1\cdot B^m)+\varphi(A^n)\varphi(B^m)\varphi(1\cdot 1) = 0,$$

and hence

$$\boldsymbol{\varphi(A^nB^m) = \varphi(A^n)\cdot\varphi(B^m)}$$

Freeness is a rule for calculating mixed moments, analogous to the concept of independence for random variables.

Thus freeness is also called **free independence**

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Note: free independence is a different rule from classical independence; free independence occurs typically for **non-commuting random variables**, like operators on Hilbert spaces or (random) matrices

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Example:

$$\varphi\Big(\big(A - \varphi(A)1\big) \cdot \big(B - \varphi(B)1\big) \cdot \big(A - \varphi(A)1\big) \cdot \big(B - \varphi(B)1\big)\Big) = 0,$$

which results in

$$\begin{aligned}\boldsymbol{\varphi(ABAB) = \varphi(AA) \cdot \varphi(B) \cdot \varphi(B) + \varphi(A) \cdot \varphi(A) \cdot \varphi(BB)} \\ \boldsymbol{- \varphi(A) \cdot \varphi(B) \cdot \varphi(A) \cdot \varphi(B)}\end{aligned}$$

Where Does Freeness Show Up?

- generators of the free group in the corresponding free group von Neumann algebras $L(\mathbb{F}_n)$

- creation and annihilation operators on full Fock spaces

- for many classes of random matrices

Understanding the Freeness Rule: the Idea of Cumulants

- write moments in terms of other quantities, which we call **free cumulants**

- freeness is much easier to describe on the level of free cumulants: **vanishing of mixed cumulants**

- relation between moments and cumulants is given by summing over **non-crossing or planar partitions**

Non-Crossing Partitions

A **partition** of $\{1,\ldots,n\}$ is a decomposition $\pi = \{V_1,\ldots,V_r\}$ with

$$V_i \neq \emptyset, \qquad V_i \cap V_j = \emptyset \quad (i \neq y), \qquad \bigcup_i V_i = \{1,\ldots,n\}$$

The V_i are the **blocks** of $\pi \in \mathcal{P}(n)$.

π is **non-crossing** if we do not have

$$p_1 < q_1 < p_2 < q_2$$

such that p_1, p_2 are in same block, q_1, q_2 are in same block, but those two blocks are different.

$$\mathbf{NC(n)} := \{\text{non-crossing partitions of } \{1,\ldots,n\}\}$$

$NC(n)$ is actually a lattice with refinement order.

Moments and Cumulants

For unital linear functional

$$\varphi : \mathcal{A} \to \mathbb{C}$$

we define **cumulant functionals** κ_n (for all $n \geq 1$)

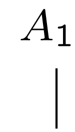
$$\kappa_n : \mathcal{A}^n \to \mathbb{C}$$

as multi-linear functionals by moment-cumulant relation

$$\varphi(A_1 \cdots A_n) = \sum_{\pi \in NC(n)} \kappa_\pi[A_1, \dots, A_n]$$

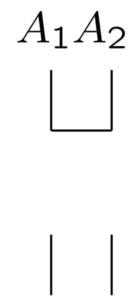
Note: classical cumulants are defined by a similar formula, where only $NC(n)$ is replaced by $\mathcal{P}(n)$

$$\varphi(A_1) = \kappa_1(A_1)$$



$$\varphi(A_1 A_2) = \kappa_2(A_1, A_2)$$

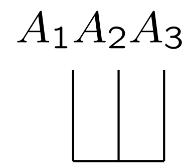
$$+ \kappa_1(A_1)\kappa_1(A_2)$$



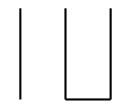
thus

$$\kappa_2(A_1, A_2) = \varphi(A_1 A_2) - \varphi(A_1)\varphi(A_2)$$

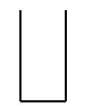
$$\varphi(A_1A_2A_3) = \kappa_3(A_1, A_2, A_3)$$



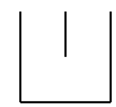
$$+\kappa_1(A_1)\kappa_2(A_2, A_3)$$



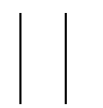
$$+\kappa_2(A_1, A_2)\kappa_1(A_3)$$



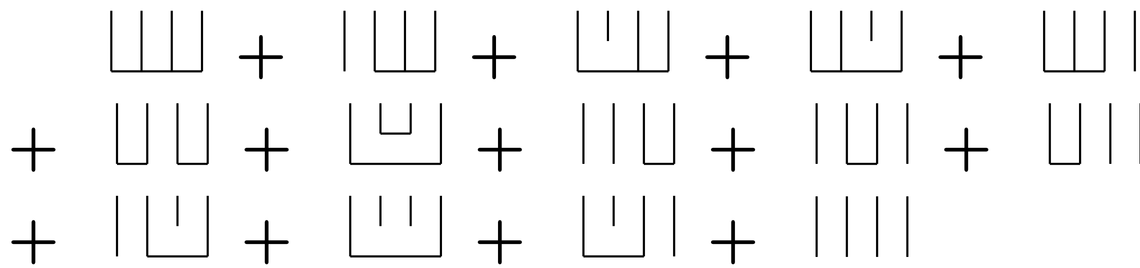
$$+\kappa_2(A_1, A_3)\kappa_1(A_2)$$



$$+\kappa_1(A_1)\kappa_1(A_2)\kappa_1(A_3)$$



$$\varphi(A_1A_2A_3A_4) =$$



$$\begin{aligned}
= \quad & \kappa_4(A_1,A_2,A_3,A_4) + \kappa_1(A_1)\kappa_3(A_2,A_3,A_4) \\
& + \kappa_1(A_2)\kappa_3(A_1,A_3,A_4) + \kappa_1(A_3)\kappa_3(A_1,A_2,A_4) \\
& + \kappa_3(A_1,A_2,A_3)\kappa_1(A_4) + \kappa_2(A_1,A_2)\kappa_2(A_3,A_4) \\
& + \kappa_2(A_1,A_4)\kappa_2(A_2,A_3) + \kappa_1(A_1)\kappa_1(A_2)\kappa_2(A_3,A_4) \\
& + \kappa_1(A_1)\kappa_2(A_2,A_3)\kappa_1(A_4) + \kappa_2(A_1,A_2)\kappa_1(A_3)\kappa_1(A_4) \\
& + \kappa_1(A_1)\kappa_2(A_2,A_4)\kappa_1(A_3) + \kappa_2(A_1,A_4)\kappa_1(A_2)\kappa_1(A_3) \\
& + \kappa_2(A_1,A_3)\kappa_1(A_2)\kappa_1(A_4) + \kappa_1(A_1)\kappa_1(A_2)\kappa_1(A_3)\kappa_1(A_4)
\end{aligned}$$

Freeness $\hat{=}$ Vanishing of Mixed Cumulants

Theorem [Speicher 1994]: The fact that A and B are free is equivalent to the fact that

$$\kappa_n(C_1, \dots, C_n) = 0$$

whenever

- $n \geq 2$
- $C_i \in \{A, B\}$ for all i
- there are i, j such that $C_i = A$, $C_j = B$

Freeness $\hat{=}$ Vanishing of Mixed Cumulants

free product $\hat{=}$ direct sum of cumulants

$\varphi(\boldsymbol{A^n})$ given by sum over **blue** planar diagrams

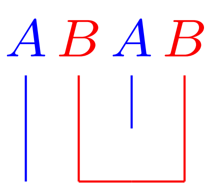
$\varphi(\boldsymbol{B^m})$ given by sum over **red** planar diagrams

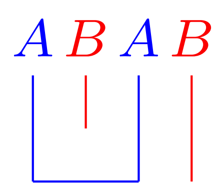
then: for A and B free, $\varphi(\boldsymbol{A^{n_1}B^{m_1}A^{n_2}}\cdots)$ is given by sum over planar diagrams with monochromatic (blue or red) blocks

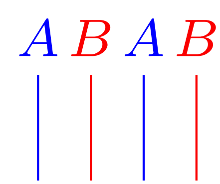
Vanishing of Mixed Cumulants

$$\varphi(ABAB) =$$

$$\kappa_1(A)\kappa_1(A)\kappa_2(B,B)+\kappa_2(A,A)\kappa_1(B)\kappa_1(B)+\kappa_1(A)\kappa_1(B)\kappa_1(A)\kappa_1(B)$$





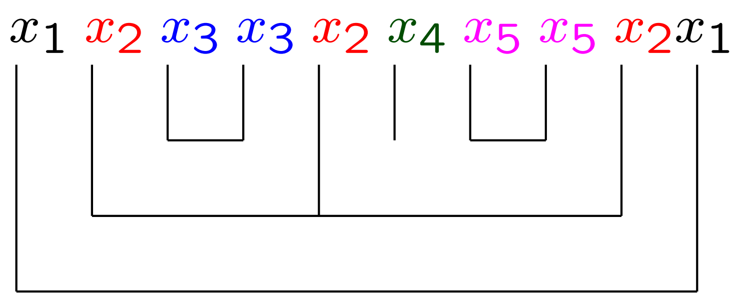


Factorization of Non-Crossing Moments

The iteration of the rule

$$\varphi(A_1BA_2) = \varphi(A_1A_2)\varphi(B) \qquad \text{if } \{A_1, A_2\} \text{ and } B \text{ free}$$

leads to the factorization of all "non-crossing" moments in free variables



$$\varphi(x_1x_2x_3x_3x_2x_4x_5x_5x_2x_1)$$

$$= \varphi(x_1x_1)\cdot\varphi(x_2x_2x_2)\cdot\varphi(x_3x_3)\cdot\varphi(x_4)\cdot\varphi(x_5x_5)$$

Sum of Free Variables

Consider A, B free.

Then, by freeness, the moments of $A+B$ are uniquely determined by the moments of A and the moments of B.

Notation: We say the distribution of $A+B$ is the

free convolution

of the distribution of A and the distribution of B,

$$\mu_{A+B} = \mu_A \boxplus \mu_B.$$

Sum of Free Variables

In principle, freeness determines this, but the concrete nature of this rule on the level of moments is not apriori clear.

Example:

$$\varphi\Big((A+B)^1\Big) = \varphi(A) + \varphi(B)$$

$$\varphi\Big((A+B)^2\Big) = \varphi(A^2) + 2\varphi(A)\varphi(B) + \varphi(B^2)$$

$$\varphi\Big((A+B)^3\Big) = \varphi(A^3) + 3\varphi(A^2)\varphi(B) + 3\varphi(A)\varphi(B^2) + \varphi(B^3)$$

$$\begin{aligned}\varphi\Big((A+B)^4\Big) = {} & \varphi(A^4) + 4\varphi(A^3)\varphi(B) + 4\varphi(A^2)\varphi(B^2) \\ & + 2\Big(\varphi(A^2)\varphi(B)\varphi(B) + \varphi(A)\varphi(A)\varphi(B^2) \\ & \quad - \varphi(A)\varphi(B)\varphi(A)\varphi(B)\Big) + 4\varphi(A)\varphi(B^3) + \varphi(B^4)\end{aligned}$$

Sum of Free Variables

Corresponding rule on level of free cumulants is easy: If A and B are free then

$$\kappa_n(A+B, A+B, \dots, A+B) = \kappa_n(A, A, \dots, A) + \kappa_n(B, B, \dots, B) + \kappa_n(\dots, A, B, \dots) + \cdots$$

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i.e., we have **additivity of cumulants for free variables**

$$\kappa_n^{A+B} = \kappa_n^A + \kappa_n^B$$

Sum of Free Variables

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$$\kappa_n^{A+B} = \kappa_n^A + \kappa_n^B$$

Also: Combinatorial relation between moments and cumulants can be rewritten easily as a relation between corresponding formal power series.

Relation between Moments and Free Cumulants

We have

$$m_n := \varphi(A^n) \qquad \text{moments}$$

and

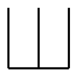
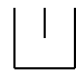
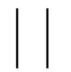
$$\kappa_n := \kappa_n(A, A, \dots, A) \qquad \text{free cumulants}$$

Combinatorially, the relation between them is given by

$$m_n = \varphi(A^n) = \sum_{\pi \in NC(n)} \kappa_\pi$$

Example:

$$m_1 = \kappa_1, \quad m_2 = \kappa_2 + \kappa_1^2, \quad m_3 = \kappa_3 + 3\kappa_2\kappa_1 + \kappa_1^3$$

$m_3 = \kappa$  $+ \kappa$  $+ \kappa$  $+ \kappa$  $+ \kappa$  $= \kappa_3 + 3\kappa_2\kappa_1 + \kappa_1^3$

Theorem [Speicher 1994]: Consider formal power series

$$M(z) = 1 + \sum_{k=1}^{\infty} m_n z^n, \qquad C(z) = 1 + \sum_{k=1}^{\infty} \kappa_n z^n$$

Then the relation

$$m_n = \sum_{\pi \in NC(n)} \kappa_\pi$$

between the coefficients is equivalent to the relation

$$M(z) = C[zM(z)]$$

Proof

First we get the following recursive relation between cumulants and moments

$$m_n = \sum_{\pi \in NC(n)} \kappa_\pi$$

$$= \sum_{s=1}^{n} \sum_{\substack{i_1,\dots,i_s \geq 0 \\ i_1+\cdots+i_s+s=n}} \sum_{\pi_1 \in NC(i_1)} \cdots \sum_{\pi_s \in NC(i_s)} \kappa_s \kappa_{\pi_1} \cdots \kappa_{\pi_s}$$

$$= \sum_{s=1}^{n} \sum_{\substack{i_1,\dots,i_s \geq 0 \\ i_1+\cdots+i_s+s=n}} \kappa_s m_{i_1} \cdots m_{i_s}$$

$$m_n = \sum_{s=1}^{n} \sum_{\substack{i_1,\dots,i_s \geq 0 \\ i_1+\dots+i_s+s=n}} \kappa_s m_{i_1} \cdots m_{i_s}$$

Plugging this into the formal power series $M(z)$ gives

$$M(z) = 1 + \sum_n m_n z^n$$

$$= 1 + \sum_n \sum_{s=1}^{n} \sum_{\substack{i_1,\dots,i_s \geq 0 \\ i_1+\dots+i_s+s=n}} k_s z^s m_{i_1} z^{i_1} \cdots m_{i_s} z^{i_s}$$

$$= 1 + \sum_{s=1}^{\infty} \kappa_s z^s \Big(M(z)\Big)^s = C[zM(z)] \qquad \square$$

Remark on Classical Cumulants

Classical cumulants c_k are combinatorially defined by

$$m_n = \sum_{\pi \in \mathcal{P}(n)} c_\pi$$

In terms of generating power series

$$\tilde{M}(z) = 1 + \sum_{n=1}^{\infty} \frac{m_n}{n!} z^n, \qquad \tilde{C}(z) = \sum_{n=1}^{\infty} \frac{c_n}{n!} z^n$$

this is equivalent to

$$\tilde{C}(z) = \log \tilde{M}(z)$$

From Moment Series to Cauchy Transform

Instead of $M(z)$ we consider **Cauchy transform**

$$G(z) := \varphi(\frac{1}{z-A}) = \int \frac{1}{z-t} d\mu_A(t) = \sum \frac{\varphi(A^n)}{z^{n+1}} = \frac{1}{z} M(1/z)$$

and instead of $C(z)$ we consider R**-transform**

$$R(z) := \sum_{n\geq 0} \kappa_{n+1} z^n = \frac{C(z)-1}{z}$$

Then $M(z) = C[zM(z)]$ becomes

$$R[G(z)] + \frac{1}{G(z)} = z \qquad \text{or} \qquad G[R(z) + 1/z] = z$$

Sum of Free Variables

Consider a random variable $A \in \mathcal{A}$ and define its **Cauchy transform G** and its **$\mathcal{R}$-transform $\mathcal{R}$** by

$$G(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\varphi(A^n)}{z^{n+1}}, \qquad \mathcal{R}(z) = \sum_{n=1}^{\infty} \kappa_n(A, \dots, A) z^{n-1}$$

Theorem [Voiculescu 1986, Speicher 1994]: Then we have

- $\frac{1}{G(z)} + \mathcal{R}(G(z)) = z$
- $\mathcal{R}^{A+B}(z) = \mathcal{R}^{A}(z) + \mathcal{R}^{B}(z)$ if A and B are free

What is Advantage of $G(z)$ over $M(z)$?

For any probability measure μ, its Cauchy transform

$$G(z) := \int \frac{1}{z-t} d\mu(t)$$

is an analytic function $G : \mathbb{C}^{+} \to \mathbb{C}^{-}$ and we can recover μ from G by **Stieltjes inversion formula**

$$d\mu(t) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \Im G(t + i\varepsilon) dt$$

Calculation of Free Convolution

The relation between Cauchy transform and $\mathcal{R}$-transform, and the Stieltjes inversion formula give an effective algorithm for calculating free convolutions; and thus also, e.g., the asymptotic eigenvalue distribution **of sums of random matrices in generic position**:

$$
\begin{array}{ccccccccc}
A & \rightsquigarrow & G^A & \rightsquigarrow & R^A & & & & \\
& & & & \downarrow & & & & \\
& & & & R^A + R^B = R^{A+B} & \rightsquigarrow & G^{A+B} & \rightsquigarrow & A+B \\
& & & & \uparrow & & & & \\
B & \rightsquigarrow & G^B & \rightsquigarrow & R^B & & & &
\end{array}
$$

What is the Free Binomial $\left(\frac{1}{2}\delta_{-1}+\frac{1}{2}\delta_{+1}\right)^{\boxplus 2}$

$$\mu := \frac{1}{2}\delta_{-1}+\frac{1}{2}\delta_{+1}, \qquad \nu := \mu \boxplus \mu$$

Then $$G_\mu(z) = \int \frac{1}{z-t}d\mu(t) = \frac{1}{2}\Big(\frac{1}{z+1}+\frac{1}{z-1}\Big) = \frac{z}{z^2-1}$$

and so $$z = G_\mu[R_\mu(z)+1/z] = \frac{R_\mu(z)+1/z}{(R_\mu(z)+1/z)^2-1}$$

thus $$R_\mu(z) = \frac{\sqrt{1+4z^2}-1}{2z}$$

and so $$R_\nu(z) = 2R_\mu(z) = \frac{\sqrt{1+4z^2}-1}{z}$$

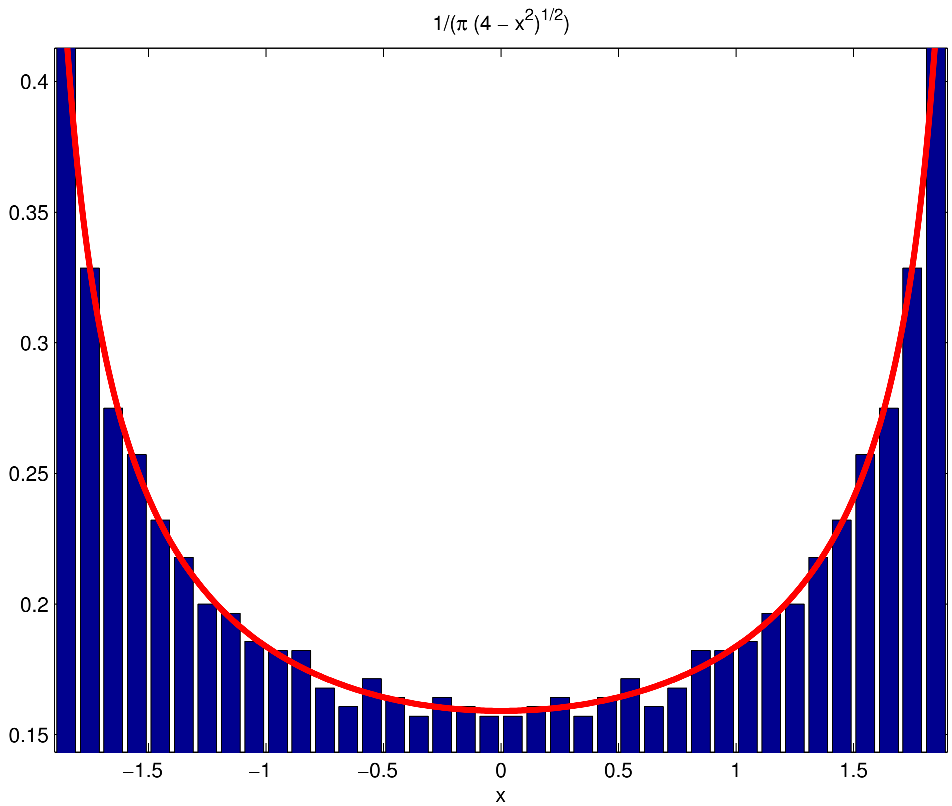
$$R_\nu(z) = \frac{\sqrt{1+4z^2}-1}{z} \qquad \text{gives} \qquad G_\nu(z) = \frac{1}{\sqrt{z^2-4}}$$

and thus

$$d\nu(t) = -\frac{1}{\pi}\Im\frac{1}{\sqrt{t^2-4}}dt = \begin{cases} \frac{1}{\pi\sqrt{4-t^2}}, & |t| \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

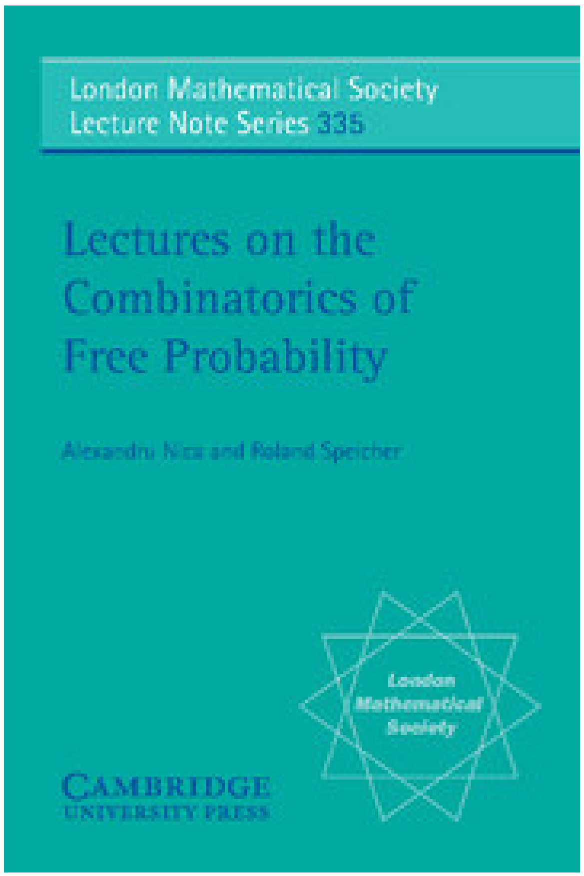
So

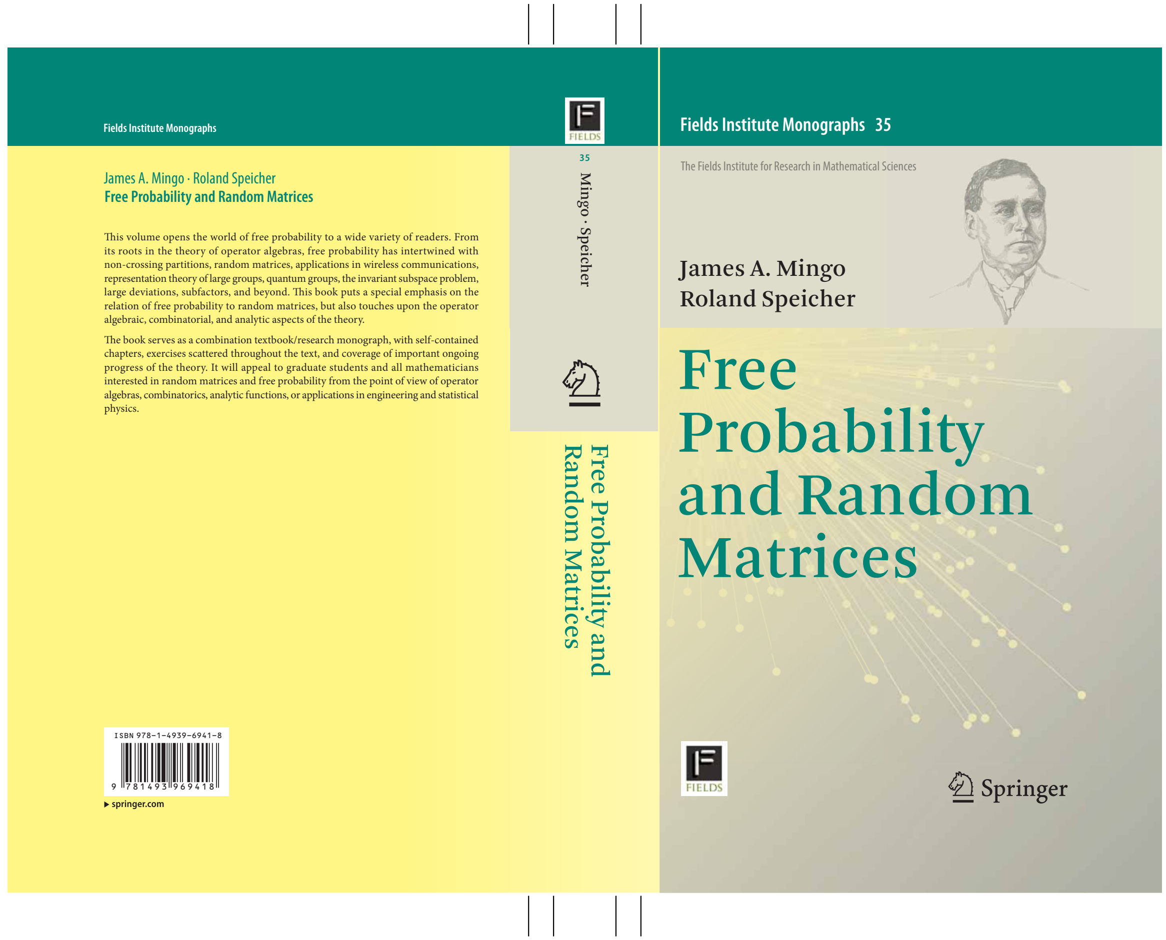
$$\Big(\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{+1}\Big)^{\boxplus 2} = \nu = \text{arcsine-distribution}$$



2800 eigenvalues of $A + UBU^*$, where A and B are diagonal matrices with 1400 eigenvalues +1 and 1400 eigenvalues -1, and U is a randomly chosen unitary matrix

Some Literature on Free Probability





Part II:

Free Brownian Motion and Free Stochastic Analysis

What is the free analogue of the normal distribution?

The free anlogue of the normal distribution is what we get as limit in a free central limit theorem: Let $x_1, x_2, \dots$ be free and identically distributed with $\varphi(x_i) = 0$ and $\varphi(x_i^2) = 1$.

To what does

$$\frac{x_1 + \cdots + x_m}{\sqrt{m}}$$

converge for $m \to \infty$?

Denote the limit by s.

We have

$$\begin{aligned}
\kappa_n(s,\dots,s) &= \lim_{m\to\infty} \kappa_n\Big(\frac{x_1+\cdots+x_m}{\sqrt{m}},\dots,\frac{x_1+\cdots+x_m}{\sqrt{m}}\Big) \\
&= \lim_{m\to\infty} \frac{1}{m^{n/2}} \sum_{i(1),\dots,i(n)=1}^{m} \kappa_n(x_{i(1)},\dots,x_{i(n)}) \\
&= \lim_{m\to\infty} \frac{1}{m^{n/2}} \sum_{i=1}^{m} \kappa_n(x_i,\dots,x_i) \\
&= \lim_{m\to\infty} m^{n/2-1} \kappa_n(x_1,\dots,x_1) \\
&= \begin{cases} 0, & n \neq 2 \\ 1, & n = 2 \end{cases}
\end{aligned}$$

μ_s has cumulants

$$\kappa_n = \begin{cases} 0, & n \neq 2 \\ 1, & n = 2 \end{cases}$$

thus

$$R(z) = \sum_{n \geq 0} \kappa_{n+1} z^n = \kappa_2 \cdot z = z$$

and hence

$$z = R[G(z)] + \frac{1}{G(z)} = G(z) + \frac{1}{G(z)}$$

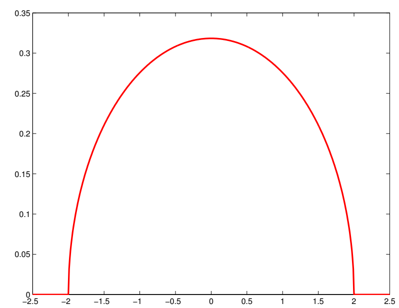
or $$G(z)^2 + 1 = zG(z)$$

$$G(z)^2 + 1 = zG(z) \qquad \text{thus} \qquad G(z) = \frac{z \pm \sqrt{z^2 - 4}}{2}$$

We have "-", because $G(z) \sim 1/z$ for $z \to \infty$; then

$$d\mu_s(t) = -\frac{1}{\pi} \Im\left(\frac{t - \sqrt{t^2 - 4}}{2}\right) dt$$

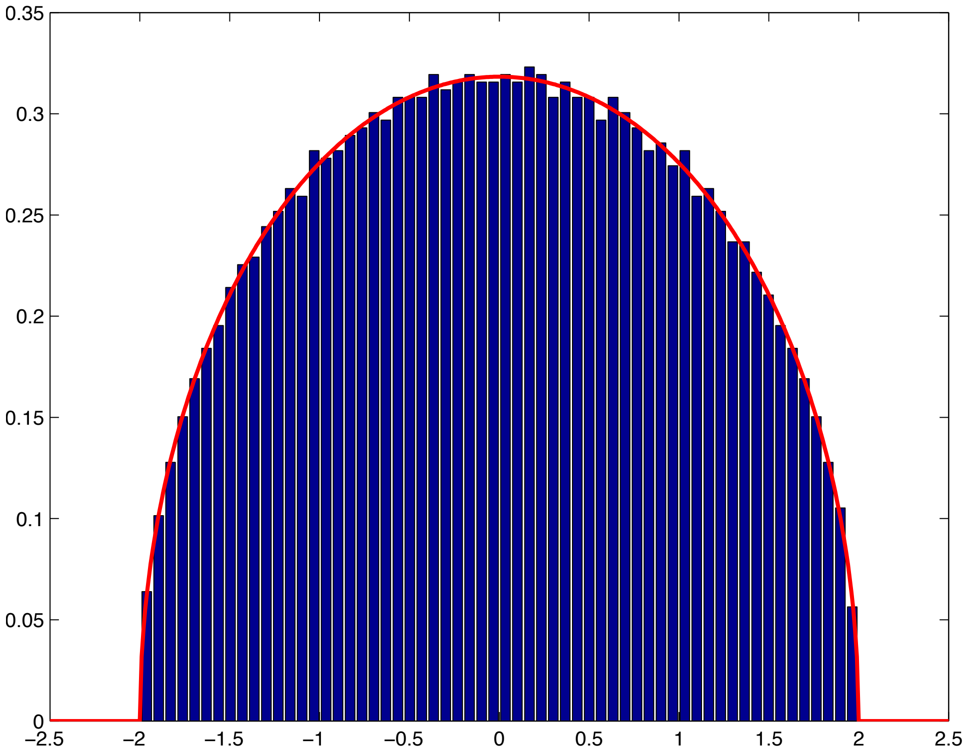
$$= \begin{cases} \frac{1}{2\pi}\sqrt{4 - t^2}\,dt, & \text{if } t \in [-2, 2] \\ 0, & \text{otherwise} \end{cases}$$



s has a **semicircular** distribution.

Wigner's semicircle law

Consider selfadjoint **Gaussian $N \times N$ random matrix**.



... one realization ...

N=4000

A **free Brownian motion** is given by a family $\big(S(t)\big)_{t\geq 0} \subset (\mathcal{A}, \varphi)$ of random variables ($\mathcal{A}$ von Neumann algebra, φ faithful trace), such that

- $S(0) = 0$

- each increment $S(t) - S(s)$ $(s < t)$ is semicircular with mean $= 0$ and variance $= t - s$, i.e.,

$$d\mu_{S(t)-S(s)}(x) = \frac{1}{2\pi(t-s)}\sqrt{4(t-s) - x^2}dx$$

- disjoint increments are free: for $0 < t_1 < t_2 < \cdots < t_n$,

$$S(t_1), \quad S(t_2) - S(t_1), \quad \ldots, \quad S(t_n) - S(t_{n-1}) \qquad \text{are free}$$

A **free Brownian motion** is given

- abstractly, by a family $\big(S(t)\big)_{t\geq 0}$ of random variables with
  - $S(0) = 0$
  - each $S(t) - S(s)$ $(s < t)$ is $(0, t-s)$-semicircular
  - disjoint increments are free
- concretely, by the sum of creation and annihilation operators on the full Fock space
- asymptotically, as the limit of matrix-valued (Dyson) Brownian motions

Free Brownian motions as matrix limits

Let $(X_N(t))_{t\geq 0}$ be a symmetric $N \times N$-matrix-valued Brownian motion, i.e.,

$$X_N(t) = \begin{pmatrix} B_{11}(t) & \dots & B_{1N}(t) \\ \vdots & \ddots & \vdots \\ B_{N1}(t) & \dots & B_{NN}(t) \end{pmatrix}, \qquad \text{where}$$

- B_{ij} are, for $i \geq j$, independent classical Brownian motions
- $B_{ij}(t) = B_{ji}(t)$.

Then, $\boldsymbol{(X_N(t))_{t\geq 0}} \xrightarrow{\text{distr}} \boldsymbol{(S(t))_{t\geq 0}}$, in the sense that almost surely

$$\lim_{N\to\infty} \text{tr}\Big(X_N(t_1)\cdots X_N(t_n)\Big) = \varphi\Big(S(t_1)\cdots S(t_n)\Big) \qquad \forall 0 \leq t_1, t_2, \dots, t_n$$

Intermezzo on realisations on Fock spaces

Classical Brownian motion can be realized quite canonically by operators on the symmetric Fock space.

Similarly, free Brownian motion can be realized quite canonically by operators on the full Fock space

First: symmetric Fock space ...

For real Hilbert space $\mathcal{H}_{\mathbb{R}}$ put $\mathcal{H} := \mathcal{H}_{\mathbb{R}} + i\mathcal{H}_{\mathbb{R}}$ and

$$\mathcal{F}_s(\mathcal{H}) := \bigoplus_{n\geq 0}^{\infty} \mathcal{H}^{\otimes \mathrm{sym} n}, \qquad \text{where} \qquad \mathcal{H}^{\otimes 0} = \mathbb{C}\Omega$$

with inner product

$$\langle f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_m \rangle_{\mathrm{sym}} = \delta_{nm} \sum_{\pi \in S_n} \prod_{i=1}^{n} \langle f_i, g_{\pi(i)} \rangle.$$

Define creation and annihilation operators (for $f \in \mathcal{H}$)

$$a^*(f) f_1 \otimes \cdots \otimes f_n = f \otimes f_1 \otimes \cdots \otimes f_n$$

$$a(f) f_1 \otimes \cdots \otimes f_n = \sum_{i=1}^{n} \langle f, f_i \rangle f_1 \otimes \cdots \otimes \check{f}_i \otimes \cdots \otimes f_n$$

$$a(f)\Omega = 0$$

... and classical Brownian motion

Put $\qquad \varphi(\cdot) := \langle \Omega, \cdot \Omega \rangle, \qquad x(f) := a(f) + a^*(f)$

then $(x(f))_{f \in \mathcal{H}_{\mathbb{R}}}$ is Gaussian family with covariance

$$\varphi\Big(x(f)x(g)\Big) = \langle f, g \rangle.$$

In particular, choose $\mathcal{H} := L^2(\mathbb{R}_+)$, $f_t := 1_{[0,t)}$, then

$$B_t := x(f_t) = a(1_{[0,t)}) + a^*(1_{[0,t)})$$

is classical Brownian motion W_t, meaning

$$\varphi(B_{t_1} \cdots B_{t_n}) = E[W_{t_1} \cdots W_{t_n}] \qquad \forall 0 \le t_1, \ldots, t_n$$

Now: full Fock space ...

For real Hilbert space $\mathcal{H}_{\mathbb{R}}$ put $\mathcal{H} := \mathcal{H}_{\mathbb{R}} + i\mathcal{H}_{\mathbb{R}}$ and

$$\mathcal{F}(\mathcal{H}) := \bigoplus_{n\geq 0}^{\infty} \mathcal{H}^{\otimes n}, \qquad \text{where} \qquad \mathcal{H}^{\otimes 0} = \mathbb{C}\Omega$$

with usual inner product

$$\langle f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_m \rangle = \delta_{nm} \langle f_1, g_1 \rangle \cdots \langle f_n, g_n \rangle.$$

Define creation and annihilation operators (for $f \in \mathcal{H}$)

$$b^*(f) f_1 \otimes \cdots \otimes f_n = f \otimes f_1 \otimes \cdots \otimes f_n$$
$$b(f) f_1 \otimes \cdots \otimes f_n = \langle f, f_1 \rangle f_2 \otimes \cdots \otimes f_n$$
$$b(f)\Omega = 0$$

... and free Brownian motion

Put $\varphi(\cdot) := \langle \Omega, \cdot \Omega \rangle, \qquad x(f) := b(f) + b^*(f)$

then $(x(f))_{f \in \mathcal{H}_\mathbb{R}}$ is semicircular family with covariance

$$\varphi\Big(x(f)x(g)\Big) = \langle f, g \rangle.$$

In particular, choose $\mathcal{H} := L^2(\mathbb{R}_+)$, $f_t := 1_{[0,t)}$, then

$S_t := x(f_t) = b(1_{[0,t)}) + b^*(1_{[0,t)})$ is free Brownian motion.

Semicircle as real part of one-sided shift

Consider case of one-dimensional $\mathcal{H} = \mathbb{C}v$. Then this reduces to

orthonormal basis for $\mathcal{F}(\mathcal{H})$: $\quad \Omega = e_0, \quad e_1, \quad e_2, \quad e_3, \quad \dots$

and one-sided shift $l = b^*(v)$

$$le_n = e_{n+1}, \qquad l^* e_n = \begin{cases} e_{n-1}, & n \geq 1 \\ 0, & n = 0 \end{cases}$$

One-sided shift is canonical non-unitary isometry

$$l^* l = 1, \qquad ll^* \neq 1 \quad (= 1 - \text{projection on } \Omega)$$

With $\varphi(a) := \langle \Omega, a\Omega \rangle$ we claim: distribution of $l + l^*$ is semicircle.

Moments of $l + l^*$

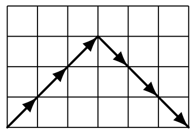
In the calculation of $\langle \Omega, (l + l^*)^n \Omega \rangle$ only such products in creation and annihilation contribute, where we never annihilate the vacuum, and where we start and end at the vacuum. So odd moments are zero.

Examples

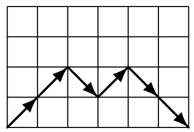
$$\varphi\Big((l + l^*)^2\Big): \quad l^* l$$
$$\varphi\Big((l + l^*)^4\Big): \quad l^* l^* l l, \quad l^* l l^* l$$
$$\varphi\Big((l + l^*)^6\Big): \quad l^* l^* l^* l l l, \quad l^* l l^* l^* l l, \quad l^* l l^* l l^* l, \quad l^* l^* l l^* l l, \quad l^* l^* l l l^* l$$

Those contributing terms are in clear bijection with non-crossing pairings (or with Dyck paths).

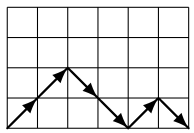
(l^*, l^*, l^*, l, l, l)



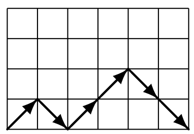
(l^*, l^*, l, l^*, l, l)



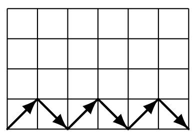
(l^*, l, l^*, l^*, l, l)



(l^*, l^*, l, l, l^*, l)



(l^*, l, l^*, l, l^*, l)



Some History on Free Stochastic Analysis

1992 stochastic integration by Kümmerer and Speicher with respect to creation and annihilation operators on full Fock space (in analogy with theory of Hudson/Pathasarathy on symmetric Fock space)

1998 improved theory of stochastic integration (and free Malliavin calculus) for free Brownian motion by Biane and Speicher

2012 free version of the fourth moment theorem by Kemp, Nourdin, Peccati, Speicher

2013 rough-paths approach to non-commutative stochastic calculus, in particular to free stochastic calculus, by Deya and Schott

Stochastic Analysis on "Wigner" space

Starting from a free Brownian motion $\Big(S(t)\Big)_{t\geq 0}$ we define multiple "Wigner" integrals

$$I(f) = \int \cdots \int f(t_1,\ldots,t_n)dS(t_1)\ldots dS(t_n)$$

for scalar-valued functions $f \in L^2(\mathbb{R}_+^n)$, by avoiding the diagonals, i.e. we understand this as

$$I(f) = \underset{\text{all } t_i \text{ distinct}}{\int \cdots \int} f(t_1,\ldots,t_n)dS(t_1)\ldots dS(t_n)$$

Definition of Wigner integrals

More precisely: for f of form

$$f = 1_{[s_1,t_1]\times\cdots\times[s_n,t_n]}$$

for pairwisely disjoint intervals $[s_1, t_1], \dots, [s_n, t_n]$ we put

$$I(f) := (S_{t_1} - S_{s_1}) \cdots (S_{t_n} - S_{s_n})$$

Extend $I(\cdot)$ linearly over set of all off-diagonal step functions (which is dense in $L^2(\mathbb{R}_+^n)$. Then observe Ito-isometry

$$\varphi[I(g)^* I(f)] = \langle f, g\rangle_{L^2(\mathbb{R}_+^n)}$$

and extend I to closure of off-diagonal step functions, i.e., to $L^2(\mathbb{R}_+^n)$.

Note: free stochastic integrals are usually bounded operators

Free Haagerup Inequality [Bozejko 1991; Biane, Speicher 1998]:

$$\Big\| \int \cdots \int f(t_1, \ldots, t_n) dS(t_1) \ldots dS(t_n) \Big\| \le (n+1) \|f\|_{L^2(\mathbb{R}^n_+)}$$

Intermezzo: combinatorics and norms

Consider free semicirculars $s_1, s_2, \dots$ of variance 1. Since $(s_1 + \cdots + s_n)/\sqrt{n}$ is again a semicircular element of variance 1 (and thus of norm 2), we have

$$\left\| \frac{1}{n^{k/2}} \sum_{i(1),\dots,i(k)=1}^{n} s_{i(1)} \cdots s_{i(k)} \right\| = \left\| \left(\frac{s_1 + \cdots + c_n}{\sqrt{n}} \right)^k \right\| = 2^k$$

The free Haagerup inequality says that this is drastically reduced if we subtract the diagonals, i.e.,

$$\lim_{n\to\infty} \left\| \frac{1}{n^{k/2}} \sum_{\substack{i(1),\dots,i(k)=1 \\ \text{all } i(.) \text{ different}}}^{n} s_{i(1)} \cdots s_{i(k)} \right\| = k + 1$$

Intermezzo: combinatorics and norms

Note: one can calculate norms from asymptotic knowledge of moments!

If x is selfadjoint and φ faithful (as our φ for the free Brownian motion is) then one has

$$\|x\| = \lim_{p\to\infty} \|x\|_p = \lim_{p\to\infty} \sqrt[p]{\varphi(|x|^p)} = \lim_{m\to\infty} \sqrt[2m]{\varphi(x^{2m})}$$

So, if s is a semicircular element, then

$$\varphi(s^{2m}) = c_m = \frac{1}{1+m}\binom{2m}{m} \sim 4^m,$$

thus

$$\|s\| = \lim_{m\to\infty} \sqrt[2m]{c_m} \sim \sqrt[2m]{4^m} = 2$$

Exercise: combinatorics and norms

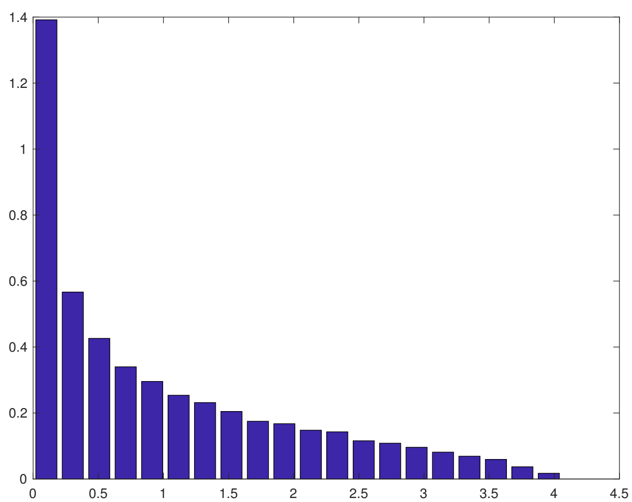
Consider free semicirculars $s_1, s_2, \ldots$ of variance 1. Prove by considering moments that

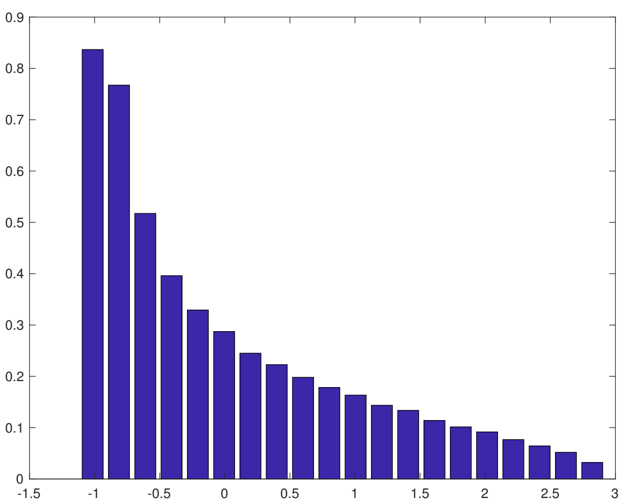
$$\left\| \frac{1}{n} \sum_{i,j=1}^{n} s_i s_j \right\| = 4, \qquad \lim_{n\to\infty} \left\| \frac{1}{n} \sum_{i,j=1, i\neq j}^{n} s_i s_j \right\| = 3$$

Realize that the involved NC pairings are in bijection with NC partitions and NC partitions without singletons, respectively.

2000 eigenvalues of the matrix

$$\frac{1}{50}\sum_{i,j=1}^{50} X_i X_j, \qquad \frac{1}{50}\sum_{i,j=1,i\neq j}^{50} X_i X_j,$$





for $X_1, \ldots, X_{50}$ independent 2000×2000 GUE.

Multiplication of Multiple Wigner Integrals

The Ito formula shows up in multiplicaton of two multiple Wigner integrals

$$\int f(t_1)dS(t_1) \cdot \int g(t_2)dS(t_2)$$

$$= \iint f(t_1)g(t_2)dS(t_1)dS(t_2) +$$

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$$= \iint f(t_1)g(t_2)dS(t_1)dS(t_2) + \int f(t)g(t)dt$$

Multiplication of Multiple Wigner Integrals

$$\iint f(t_1, t_2)dS(t_1)dS(t_2) \cdot \int g(t_3)dS(t_3)$$

$$= \iiint f(t_1, t_2)g(t_3)dS(t_1)dS(t_2)dS(t_3)$$

$$+ \iint f(t_1, t)g(t)dS(t_1)\underbrace{dS(t)dS(t)}_{dt}$$

$$+ \iint f(t, t_2)g(t)\underbrace{dS(t)dS(t_2)dS(t)}$$

Multiplication of Multiple Wigner Integrals

$$\iint f(t_1,t_2)dS(t_1)dS(t_2) \cdot \int g(t_3)dS(t_3)$$

$$= \iiint f(t_1,t_2)g(t_3)dS(t_1)dS(t_2)dS(t_3)$$

$$+ \iint f(t_1,t)g(t)dS(t_1)\underbrace{dS(t)dS(t)}_{dt}$$

$$+ \iint f(t,t_2)g(t)\underbrace{dS(t)dS(t_2)dS(t)}_{dt\varphi[dS(t_2)]=0}$$

Multiplication of Multiple Wigner Integrals

Free Ito Formula [Biane, Speicher 1998]:

$$dS(t)AdS(t) = \varphi(A)dt \quad \text{for } A \text{ adapted}$$

Multiplication of Multiple Wigner Integrals

Consider $f \in L^2(\mathbb{R}_+^n)$, $\qquad g \in L^2(\mathbb{R}_+^m)$

For $0 \leq p \leq \min(n,m)$, define

$$f \overset{p}{\frown} g \in L^2(\mathbb{R}_+^{n+m-2p})$$

by

$$f \overset{p}{\frown} g(t_1, \ldots, t_{m+n-2p})$$

$$= \int f(t_1, \ldots, t_{n-p}, s_1, \ldots, s_p) g(s_p, \ldots, s_1, t_{n-p+1}, \ldots, t_{n+m-2p}) ds_1 \cdots ds_p$$

Then we have

$$I(f) \cdot I(g) = \sum_{p=0}^{\min(n,m)} I(f \overset{p}{\frown} g)$$

Example: $f \in L^2(\mathbb{R}_+^3)$, $g \in L^2(\mathbb{R}_+^4)$

$$I(f) \cdot I(g)$$

$$= \int f(t_1, t_2, t_3) dS(t_1) dS(t_2) dS(t_3) \cdot \int g(s_1, s_2, s_3, s_4) dS(s_1) \ldots dS(s_4)$$

$$= \bigcirc \bigcirc \bigcirc \mid \bigcirc \bigcirc \bigcirc \bigcirc$$

$$= \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc + \bigcirc \bigcirc \circ \circ \bigcirc \bigcirc \bigcirc$$

$$+ \bigcirc \circ \circ \circ \circ \bigcirc \bigcirc + \circ \circ \circ \circ \circ \circ \bigcirc$$

$$= I(f \overset{0}{\frown} g) + I(f \overset{1}{\frown} g) + I(f \overset{2}{\frown} g) + I(f \overset{3}{\frown} g)$$

$$\int 1_{[0,1]}(t) dS(t) = S(1) - S(0) = S \qquad \text{semicircular variable}$$

What is

$$U_n := \int 1_{[0,1]^n}(t_1, \dots, t_n) dS(t_1) \cdots dS(t_n)$$

We have

$$S \cdot U_n = \bigcirc \mid \bigcirc \bigcirc \cdots \bigcirc = \bigcirc \bigcirc \bigcirc \cdots \bigcirc + \circ \circ \bigcirc \cdots \bigcirc$$

$$= U_{n+1} + U_{n-1}$$

Thus

$$S \cdot U_n = U_{n+1} + U_{n-1} \qquad \text{recursion for Chebycheff polynomials}$$

$$U_1 = S, \quad U_2 = S^2 - 1, \qquad U_3 = S^3 - 2S, \qquad \dots$$

Note

$$U_n = S^n + \text{smaller degree polynomial}$$

In this special case, Haagerup inequality is saying that

$$\|S^n\| = 2^n$$

is reduced to

$$\|U_n\| = n + 1$$

For example,

$$\begin{array}{ccc} \|S\| = 2, & \|S^2\| = 4, & \|S^3\| = 8 \\ \|S\| = 2, & \|S^2 - 1\| = 3, & \|S^3 - 2S\| = 4 \end{array}$$

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$$\|S\| = 2, \qquad \|S^2 - 1\| = 3, \qquad \|S^3 - 2S\| = 4$$

This follows here from

$$\|U_n\| = \sup_{|t|\le 2} |U_n(t)| \qquad U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}$$

Compare to classical analogue

$$\int 1_{[0,1]}(t)dB(t) = B(1) - B(0) = N \qquad \text{normal variable}$$

$$H_n := \int 1_{[0,1]^n}(t_1, \dots, t_n)dB(t_1) \cdots dB(t_n)$$

We have

$$N \cdot H_n = \bigcirc \mid \bigcirc \bigcirc \cdots \bigcirc$$

$$= \bigcirc \bigcirc \bigcirc \cdots \bigcirc + \circ \circ \bigcirc \cdots \bigcirc + \circ \bigcirc \circ \cdots \bigcirc + \circ \bigcirc \bigcirc \cdots \circ$$

$$= H_{n+1} + nH_{n-1}$$

$$N \cdot H_n = H_{n+1} + nH_{n-1} \qquad \text{recursion for Hermite polynomials}$$

$$H_1 = N, \qquad H_2 = N^2 - 1, \qquad H_3 = N^3 - 3N, \qquad \dots$$

Note that for $n \geq 1$:

$$\varphi[I(f)] = \int f(t_1, \ldots, t_n)\varphi[dS(t_1) \cdots dS(t_n)] = 0$$

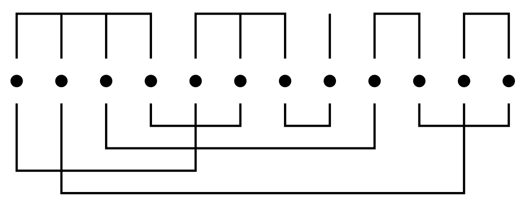
Thus, for $f_i \in L^2(\mathbb{R}_+^{n_i})$

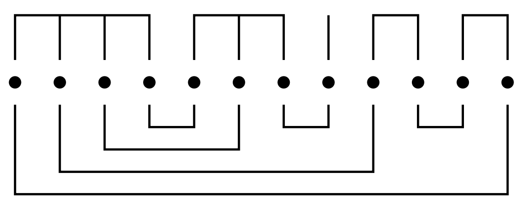
$$\varphi[I(f_1) \cdots I(f_r)] = \text{only terms with total contractions}$$

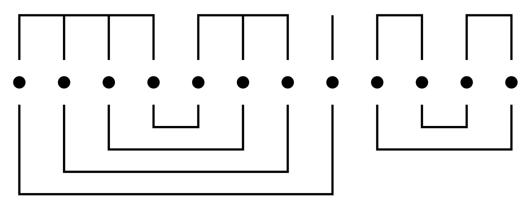
$$= \sum_{\pi \in NC_2(n_1 \otimes \cdots \otimes n_r)} \int_\pi f_1 \otimes \cdots \otimes f_r$$

Example: $I(f_1)I(f_2)I(f_3)I(f_4)I(f_5)$ with

$$(n_1, n_2, n_3, n_4, n_5) = (4, 3, 1, 2, 2)$$







The first picture contributes in the classical case, but not in the free case. The other two pictures contribute here.

More general: all $\pi \in NC(n_1 + \cdots + n_r)$ contribute with

$$\pi \wedge \{[n_1], [n_1, n_1 + n_2], \ldots\} = 0$$

In particular, we have for $f, g \in L^2(\mathbb{R}_+^n)$

$$\begin{aligned}
&\varphi[I(f)I(g)^*] \\
&= \int f(t_1, \ldots, f_n)\bar{g}(s_1, \ldots, s_n)\varphi[dS(t_1)\ldots dS(t_n) \cdot dS(s_n)\ldots dS(s_1)] \\
&= \int f(t_1, \ldots, t_n)\bar{g}(t_1, \ldots, t_n)dt_1 \ldots dt_n \\
&= \langle f, g\rangle_{L^2(\mathbb{R}_+^n)}
\end{aligned}$$

and for $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$ with $n \neq m$

$$\varphi[I(g)I(g)^*] = 0$$

or more general, for $f, g \in \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}_+^n)$

$$\varphi[I(f)I(g)^*] = \langle f, g\rangle$$

Free Chaos Decomposition

One has the canonical isomorphism

$$L^2(\{S(t) \mid t \geq 0\}) \quad \widehat{=} \quad \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}_+^n), \qquad f \quad \widehat{=} \quad \bigoplus_{n=0}^{\infty} f_n,$$

via

$$f = \sum_{n=0}^{\infty} I(f_n) = \sum_{n=0}^{\infty} \int \cdots \int f_n(t_1, \dots, t_n) dS(t_1) \dots dS(t_n).$$

The set

$$\{I(f_n) \mid f_n \in L^2(\mathbb{R}_+^n)\}$$

is called **n-th (Wigner) chaos**.

Theorem [Kemp, Nourdin, Peccati, Speicher 2012]: Consider, for fixed n, a sequence $f_1, f_2, \dots \in L^2(\mathbb{R}_+^n)$ with $f_k^* = f_k$ and $\|f_k\|_2 = 1$ for all $k \in \mathbb{N}$. Then the following statements are equivalent.

(i) We have $\lim_{k\to\infty} \varphi[I(f_k)^4] = 2$.

(ii) We have for all $p = 1, 2, \dots, n-1$ that

$$\lim_{k\to\infty} f_k \overset{p}{\frown} f_k = 0 \qquad \text{in } L^2(\mathbb{R}_+^{2n-2p}).$$

(iii) The selfadjoint variable $I(f_k)$ converges in distribution to a semicircular variable of variance 1.

Corollary: For $n \geq 2$ and $f \in L^2(\mathbb{R}_+^n)$, the law of $I(f)$ is not semicircular

Thus for $n \neq 2$

$$\{\text{distributions in first chaos}\} \cap \{\text{distribution in } n\text{-th chaos}\} = \emptyset$$

The more general question for $n \neq m$

$\{$distributions in m-th chaos$\}\cap\{$distribution in n-th chaos$\}$ =???

is still open.

[For the classical case one knows that all Wiener chaoses have disjoint distributions.]

Idea of proof of 4^{th} moment theorem

- $I(f_k) \to s$ implies $\varphi[I(f_k)^4] \to \varphi[s^4] = 2$ is clear

- for the other direction first show: convergence of fourth moment implies vanishing of non-trivial contractions

- then show: if non-trivial contractions are zero, then all moments calculate as the moments for a semicircular

Vanishing of contractions

$$I(f)^* = I(f^*) \qquad \text{where} \qquad f^*(t_1, \dots, f_n) := \overline{f(t_n, \dots, t_1)}$$

Then

$$I(f)I(f^*) = \sum_{p=0}^{n} I(f \overset{p}{\frown} f^*)$$

Note: $f \overset{p}{\frown} f^* \in L^2(\mathbb{R}_+^{2n-2p})$, thus terms for different p are orthogonal in L^2 and thus

$$\varphi[|I(f)|^4] = \varphi[(I(f)I(f^*))^2] = \sum_{p=0}^{n} \varphi[I(f \overset{p}{\frown} f^*)^2]$$

But for the two trivial contractions $p=0$ and $p=n$ we have

$I(f \overset{0}{\frown} f^*) = I(f \otimes f^*)$, thus $\varphi[I(f \overset{0}{\frown} f^*)^2] = \|f \otimes f^*\|_{L^2}^2 = 1$

and

$$f \overset{n}{\frown} f^* = \|f\|^2 = 1, \quad \text{thus} \quad \varphi[I(f \overset{n}{\frown} f^*)^2] = 1$$

and so

$$\varphi[|I(f)|^4] = \sum_{p=0}^{n} \varphi[I(f \overset{p}{\frown} f^*)^2] = 2 + \sum_{p=1}^{n-1} \varphi[I(f \overset{p}{\frown} f^*)^2]$$

Thus: if $\varphi[|I(f)|^4] = 2$, then

$$f \overset{p}{\frown} f^* = 0 \qquad \text{for all } p = 1, \dots, n-1$$

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Thus: if $\varphi[|I(f)|^4] = 2$, then

$$f \overset{p}{\frown} f^* = 0 \qquad \text{for all } p = 1, \dots, n-1$$

[Note: all this cannot happen for one f if $n > 1$, but all arguments are also valid in the limit $k \to \infty$ for sequences f_k]

Calculation of higher moments

The vanishing of non-trivial contractions implies that in the calculation of

$$\varphi[I(f)^r] = \sum_{\pi \in NC_2(n^{\otimes r})} \int_\pi f^{\otimes r}$$

contractions between p arguments of two of the involved f must be trivial, i.e, either no contractions at all ($p = 0$), or a total contraction ($p = n$)

But such contractions are in bijection with non-crossing pairings of r elements, i.e., we get

$$\varphi[I(f)^r] = \#NC_2(r) = \varphi(s^r)$$

$\square$

Results on Regularity of Distributions

Theorem [Shlyakhtenko, Skoufranis 2013; Mai, Speicher, Weber 2014]: Let p be a non-constant selfadjoint polynomial and $s_1, \dots, s_n$ free semicirculars. Then the distribution of $p(s_1, \dots, s_n)$ does not have atoms.

Theorem [Mai 2015]: The distribution of a non-constant finite selfadjoint Wigner integral

$$\sum_{k=1}^{n} \int f_k(t_1, \dots, t_k) dS(t_1,) \cdots dS(t_k)$$

does not have atoms.

Idea of Proofs

The results of Mai, Speicher, Weber rely on having a calculus of non-commutative derivatives for our polynomials:

- having atoms for some polynomials implies by differentiation that one also has atoms for the derivative
- but then, by iteration, one should have atoms for linear polynomials
- which is not the case

The result of Mai on stochastic integrals relies on a version of such a differential calculus in the setting of stochastic integrals ... this is the free Malliavin calculus

Some literature on free stochastic analysis

Biane, Speicher: Stochastic Calculus with Respect to Free Brownian Motion and Analysis on Wigner Space. *Prob. Theory Rel. Fields*, 1998

Kemp, Nourdin, Peccati, Speicher: Wigner Chaos and the Fourth Moment. *Ann. Prob.*, 2012

Nourdin, Peccati, Speicher: Multidimensional Semicircular Limits on the Free Wigner Chaos. *Seminar on Stochastic Analysis, Random Fields and Applications VII*, 2013

Afterword on Quantitative Estimates ...

Given two self-adjoint random variables X, Y, define the distance

$$d_{\mathcal{C}_2}(X,Y) := \sup\{|\varphi[h(X)] - \varphi[h(Y)]| \,:\, \mathcal{I}_2(h) \le 1\};$$

where

$$\mathcal{I}_2(h) \quad \widehat{=} \quad \|\partial h'\| \qquad \text{and} \qquad \partial X^n = \sum_{k=0}^{n-1} X^k \otimes X^{n-1-k}$$

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Rigorously: If h is the Fourier transform of a complex measure ν on $\mathbb{R}$,

$$h(x) = \widehat{\nu}(x) = \int_{\mathbb{R}} e^{ix\xi}\,\nu(d\xi)$$

then we define

$$\mathcal{I}_2(h) = \int_{\mathbb{R}} \xi^2\,|\nu|(d\xi)$$

... in Terms of the Free Gradient Operator

Define the free Malliavin gradient operator by

$$\nabla_t \left(\int f(t_1, \ldots, t_n)\, dS_{t_1} \cdots dS_{t_n} \right)$$
$$:= \sum_{k=1}^{n} \int f(t_1, \ldots, t_{k-1}, t, t_{k+1}, \ldots, t_n)$$
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Theorem [Kemp, Nourdin, Peccati, Speicher 2012]:

$$d_{\mathcal{C}_2}(F, S) \leq \frac{1}{2} \varphi \otimes \varphi \left(\left| \int \nabla_s (N^{-1} F) \sharp (\nabla_s F)^* \, ds - 1 \otimes 1 \right| \right)$$

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But no estimate against Fourth Moment in this case!

In the classical case one can estimate the corresponding expression of the above gradient, for F living in some n-th chaos, in terms of the fourth moment of the considered variable, thus giving a quantitative estimate for the distance between the considered variable (from a fixed chaos) and a normal variable in terms of the difference between their fourth moments. In the free case such a general estimate does not seem to exist; at the moment we are only able to do this for elements F from the second chaos.

Corollary: Let $F = I(f) = I(f)^*$ $(f \in L^2(\mathbb{R}_+^2))$ be an element from the second chaos with variance 1, i.e., $\|f\|_2 = 1$, and let S be a semiciruclar variable with mean 0 and variance 1. Then we have

$$d_{\mathcal{C}_2}(F, S) \leq \frac{1}{2}\sqrt{\frac{3}{2}}\sqrt{\varphi(F^4) - 2}.$$

Some general literature

- D. Voiculescu, K. Dykema, A. Nica: Free Random Variables. CRM Monograph Series, Vol. 1, AMS 1992

- F. Hiai, D. Petz: The Semicircle Law, Free Random Variables and Entropy. Math. Surveys and Monogr. 77, AMS 2000

- A. Nica, R. Speicher: Lectures on the Combinatorics of Free Probability. *London Mathematical Society Lecture Note Series*, vol. 335, Cambridge University Press, 2006

- J. Mingo, R. Speicher: Free Probability and Random Matrices. *Fields Institute Monographs*, vol. 35, Springer, 2017