Rough Path Approach to Non-commutative Stochastic Processes: Free Brownian Motion and q-Brownian Motion

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Section 1

Free Brownian Motion and Its Stochastic Calculus

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Free Brownian Motion

Definition

A **free Brownian motion** is given by a family $(S_t)_{t\geq 0} \subset (\mathcal{A}, \varphi)$ of random variables ($\mathcal{A}$ von Neumann algebra, φ faithful trace), such that

- $S_0 = 0$
- each increment $S_t - S_s$ $(s < t)$ is semicircular with mean $= 0$ and variance $= t - s$, i.e.,

$$d\mu_{S_t - S_s}(x) = \frac{1}{2\pi(t-s)}\sqrt{4(t-s) - x^2}dx$$

- disjoint increments are free: for $0 < t_1 < t_2 < \cdots < t_n$,

$$S_{t_1}, \quad S_{t_2} - S_{t_1}, \quad \ldots, \quad S_{t_n} - S_{t_{n-1}} \qquad \text{are free}$$

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Free Stochastic Calculus

History

- Kümmerer + Speicher: JFA 1992
- Biane + Speicher: PTRF 1998

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Goal

For processes

$$(A_t)_{t\geq 0}, \quad (B_t)_{t\geq 0} \quad \subset \mathcal{A}$$

(functions of the free Brownian motion) define

$$\int A_t dS_t B_t.$$

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Ito-Type Definition for Adapted Processes

As usual: processes must be adapted

Definition

$(A_t)_{t\geq 0}$ is **adapted** if

$$A_t \in \mathrm{vN}\big(S(\tau) \mid \tau \leq t\big) \qquad \forall t \geq 0$$

Definition

Then define for piecewise constant processes

$$\int A_t dS_t B_t := \sum_i A_{t_i} \big(S_{t_{i+1}} - S_{t_i}\big) B_{t_i}$$

and extend by continuity

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Norm Estimates for Free Stochastic Integrals

- **Ito isometry:** for the L^2 norm $\|a\|_2^2 := \varphi(aa^*)$ we have

$$\| \int A_t dS_t B_t \|_2^2 = \int \|A_t\|_2^2 \cdot \|B_t\|_2^2 dt$$

note: this is essentially the fact that for a semicircle S of variance dt, which is free from $\{a, a^*, b, b^*\}$ we have

$$\varphi(aSbb^*Sa^*) = \varphi(bb^*)\varphi(aa^*)dt$$

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- **free Burkholder-Gundy inequality for $p = \infty$:** for the operator norm we have the much deeper estimate

$$\| \int A_t dS_t B_t \|^2 \leq c \cdot \int \|A_t\|^2 \cdot \|B_t\|^2 dt$$

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Free Ito Formula

We have as for classical Brownian motion

$$dS_t dS_t = dt$$

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Free Ito Formula

We have as for classical Brownian motion

$$dS_t dS_t = dt$$

... but that's not all, we also need

$$dS_t A dS_t = \varphi(A) dt$$

for A adapted
[Classically we have of course: $dW_t A dW_t = A dt$]

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Section 2

q-Brownian Motion and Its Stochastic Calculus

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q-Brownian Motion

Let $(W_t)_{t\geq 0}$ be classical Brownian motion and $(S_t)_{t\geq 0}$ be free Brownian motion, then we have for their joint moments the "Wick formula" with covariance function $c(t,s) = \min(t,s)$.

$$E[W_{t_1} \cdots W_{t_n}] = \sum_{\pi \in \mathcal{P}_2(n)} \prod_{(i,j)\in\pi} c(t_i, t_j)$$

$$\varphi[S_{t_1} \cdots S_{t_n}] = \sum_{\pi \in NC_2(n)} \prod_{(i,j)\in\pi} c(t_i, t_j)$$

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Definition (Bożejko, Speicher '91; Bożejko, Kümmerer, Speicher '97)

For $0 \leq q \leq 1$ we define a **q-Brownian motion** $(X_t)_{t\geq 0}$ by the following q-version of a Wick formula

$$\varphi[X_{t_1} \cdots X_{t_n}] = \sum_{\pi \in \mathcal{P}_2(n)} q^{\text{crossings of } \pi} \prod_{(i,j)\in\pi} c(t_i, t_j)$$

q-Stochastic Calculus

- Donati-Martin 2003
  Definition of Ito-type stochastic integral

$$\int A_t dX_t B_t \qquad \text{for adapted } A, B$$

  in L^2-setting

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$$dX_t dX_{t_1} \cdots dX_{t_n} dX_t = q^n dX_{t_1} \cdots dX_{t_n} dt$$

  for $t_i \neq t_j$ $(i \neq j)$

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- Deya, Schott 2017
  Definition of Stratonovich-type stochastic integrals in L^∞-setting via rough path approach

Section 3

Rough Path Approach to Non-Commutative Stochastic Integration

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- **geometric rough path** given by geometric Levy area

$$\mathbb{X}_{st} \mathrel{\hat{=}} \int_{s\le t_1\le t_2\le t} dX_{t_1} \otimes dX_{t_2}$$

  - Capitaine, Donati-Martin 2001; Victoir 2004: for free BM
  - Deya, Schott 2013: for q-case

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- **geometric case:** allows to deal with one-sided SDE

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- controlled case: allows to deal with two-sided SDE

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- **controlled rough path** theory
  - Deya, Schott 2013: for free BM
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Recall commutative case for $1/3 \leq \alpha < 1/2$:

We want to define

$$I(t) = \int_0^t f_\tau dX_\tau, \qquad \text{or} \qquad (\delta_1 I)_{st} = \int_s^t f_\tau dX_\tau$$

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Recall commutative case for $1/3 \leq \alpha < 1/2$:

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For this we use approximating germ

$$A_{st} = f_s(X_t - X_s) + f'_s Y_{st}$$

where quadratic correction Y should satisfy

$$(\delta_2 Y)_{sut} = (X_u - X_s)(X_t - X_u)$$

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$$(\delta_2 Y)_{sut} = (X_u - X_s)(X_t - X_u) \qquad \textbf{just take: } Y_{st} = \frac{1}{2}(X_t - X_s)^2$$

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1 non-commutative dimension $\hat{=}$ ∞-many commutative dimensions

Hence this explicit Y_{st} cannot be used as quadratic (Stratonovic) correction. One has to define

$$Y_{st} \hat{=} \int_{s \leq t_1 \leq t_2 \leq t} dX_{t_1} dX_{t_2}$$

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by some other means. Actually, we need more general a **product Levy area** of the form

$$Y_{st}[A \otimes B] \hat{=} \int_{s \leq t_1 \leq t_2 \leq t} A dX_{t_1} B dX_{t_2};$$

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formally this is a family $(Y_{st}[A \otimes B])_{s \le t}$ which satisfies

- some adequate L^∞-regularity
- Chen identity

$$Y_{st}[A \otimes B] - Y_{su}[A \otimes B] - Y_{ut}[A \otimes B] = A(X_u - X_s)B(X_t - X_u)$$

Results for Free and q-Brownian Motion

The definition of such a product Levy area

$$Y_{st}[A \otimes B] \hat{=} \int_{s \le t_1 \le t_2 \le t} A dX_{t_1} B dX_{t_2}$$

(and thus of a non-commutative stochastic integration theory) was achieved

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- for the q-Brownian motion in Deya, Schott 2017: by defining the product Levy area directly, via heavy L^p-calculations for $p \to \infty$, as a limit of linear interpolations via dyadic partitions

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