Rough paths, regularity structures and renormalisation

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Integration and Multiplication

Let $f, g : [0, T] \to \mathbb{R}$ two continuous functions.

What does it mean to define the integral

$$\int_0^T f_r \, \dot{g}_r \, \mathrm{d}r$$

when f, g are not differentiable ?

Important example: $g = B$ with $(B_t)_{t \geq 0}$ a Brownian motion.

Starting point of the Rough Paths theory (Terry Lyons, Massimiliano Gubinelli).

Example of a more general problem: given a distribution ($\dot{g}$) and a non-smooth function (f), how can we define their product? Namely a distribution $f\dot{g}$.

Local approximation

If g is of class C^1, then we define

$$I_t := \int_0^t f_r \, \dot{g}_r \, \mathrm{d}r, \qquad t \in [0,T].$$

Then we have $I_0 = 0$ and for $0 \le s \le t \le T$

$$I_t - I_s - f_s(g_t - g_s) = \int_s^t (f_r - f_s) \, \dot{g}_r \, \mathrm{d}r = o(|t-s|).$$

We write

$$I_0 = 0, \qquad I_t - I_s = f_s(g_t - g_s) + R_{st}, \qquad R_{st} = o(|t-s|).$$

These properties characterise $(I_t)_{t\in[0,T]}$, since if we have I^1 and I^2 then setting $I^{12} := I^1 - I^2$

$$|I_t^{12} - I_s^{12}| = o(|t-s|)$$

which implies I^{12} constant.

Local approximation

Let us still study the formula

$$I_0 = 0, \qquad I_t - I_s = f_s(g_t - g_s) + R_{st}, \qquad R_{st} = o(|t-s|).$$

If we compute for $0 \le s \le u \le t \le T$

$$R_{st} - R_{su} - R_{ut} = (f_u - f_s)(g_t - g_u)$$

which does not depend on I.

Therefore the existence of I is equivalent to the existence of R such that the above formula holds.

A cochain complex

Les us define for $n \geq 1$

$$\Delta_n := \{(t_1, \dots, t_n) \in [0, T]^n : t_1 \leq \cdots \leq t_n\},$$

$$C_n := \{f : \Delta_n \to \mathbb{R} \quad \text{continuous}\},$$

$$\delta_n : C_n \to C_{n+1}, \qquad (\delta_n f)_{t_1 \dots t_{n+1}} = \sum_{k=1}^{n+1} (-1)^{n+2-k} f_{t_1 \dots \hat{t}_k \dots t_{n+1}}.$$

Then we have

- $\delta_{n+1} \circ \delta_n \equiv 0$ (exercise!)
- if $g \in C_{n+1}$ and $\delta_{n+1}\, g = 0$, then $g = \delta_n f$ with $f \in C_n$ (exercise!).

In particular we have an exact cochain complex

$$\mathbb{R} \to C_1 \xrightarrow{\delta_1} C_2 \xrightarrow{\delta_2} C_3 \xrightarrow{\delta_3} \cdots$$

Local approximation

Therefore, existence of $I \in C_1$ such that

- $I_0 = 0$,
- $(\delta_1 I)_{st} = f_s(g_t - g_s) + o(|t-s|)$, where $(\delta_1 I)_{st} = I_t - I_s$,

is equivalent to the existence of $R \in C_2$ such that

- $(\delta_2 R)_{sut} = (f_u - f_s)(g_t - g_u)$, where $(\delta_2 R)_{sut} = R_{st} - R_{su} - R_{ut}$,
- $R_{st} = o(|t-s|)$.

Gubinelli calls I the integral, $A_{st} := f_s(g_t - g_s)$ the germ, and R_{st} the remainder.

The sewing lemma

For $\alpha > 0$ and $h \in C_n$ we set

$$\|h\|_\alpha := \sup_{(t_1,\dots,t_n)\in\Delta_n} \frac{|h(t_1,\dots,t_n)|}{|t_n - t_1|^\alpha}$$

and we say that $h \in C_n^\alpha$ if $\|h\|_\alpha < +\infty$. We also set $C_n^{\alpha+} := \cup_{\beta>\alpha} C_n^\beta$.

Theorem (Gubinelli)

There exists a unique map $\Lambda : C_3^{1+} \cap \delta_2 C_2 \to C_2^{1+}$ *such that* $\delta_2 \Lambda = \mathrm{id}_{C_3^{1+}\cap\delta_2 C_2}$. *Moreover* Λ *satifies for all* $\alpha > 1$

$$\|\Lambda B\|_\alpha \le K_\alpha \|B\|_\alpha, \qquad B \in C_3^{1+} \cap \delta_2 C_2.$$

Proof.

See the first lecture sheet of ▸ MG □

A first application: Young integration

Theorem

If $f \in C^\alpha$, $g \in C^\beta$ *(standard* Hölder *spaces) with* $\alpha + \beta > 1$ *then there exists a unique pair* $(I, R) \in C^\beta \times C_2^{\alpha+\beta}$ *such that*

$$I_0 = 0, \qquad I_t - I_s = f_s(g_t - g_s) + R_{st}.$$

The map

$$C^\alpha \times C^\beta \ni (f, g) \to I \in C^\beta$$

is the unique continuous extension of

$$C^1 \times C^1 \ni (f, g) \to \int_0^\bullet f\, \dot{g}\, \mathrm{d}u \in C^1.$$

Proof

- Existence. Setting $A_{st} := f_s(g_t - g_s) \in C_2^\beta$, we already know that $(\delta_2 A)_{sut} = -(f_u - f_s)(g_t - g_u)$, $0 \le s \le t \le T$, so that

$$|(\delta_2 A)_{sut}| \le C\,|u-s|^\alpha |t-u|^\beta \le C\,|t-s|^{\alpha+\beta}.$$

  Setting $R := -\Lambda\delta_2 A \in C_2^{\alpha+\beta}$ then $A + R \in C_2^\beta$ and $\delta_2(A+R) = \delta_2 A - \delta_2\Lambda\delta_2 A = 0$, so that $A + R = \delta_1 I$ with $I \in C^\beta$.
- Uniqueness. If I^1, I^2 then $|I^{12}_t - I^{12}_s| = o(|t-s|)$.
- Continuity. The estimate

$$\|I\|_{C^\beta} \lesssim \|f\|_{C^\alpha}\|g\|_{C^\beta}$$

  follows from

$$\|\Lambda\delta_2 A\|_{\alpha+\beta} \le K_{\alpha+\beta}\|\delta_2 A\|_{\alpha+\beta}, \qquad \delta_2 A \in C_3^{\alpha+\beta} \cap \delta_2 C_2.$$

  in the Sewing Lemma.

Dyadic approximation

Let us consider for $t_i^n := i2^{-n}T$ and $n \geq 0$

$$I_t^n = \sum_{i=1}^{2^n} \mathbb{1}_{(t_i^n \leq t)} A_{t_{i-1}^n t_i^n}.$$

Then, since $t_{2i}^{n+1} = t_i^n$,

$$\begin{aligned}|I_t^n - I_t^{n+1}| &= \left| \sum_{i=1}^{2^n} \mathbb{1}_{(t_i^n \leq t)} \left(A_{t_{i-1}^n t_i^n} - A_{t_{2i-2}^{n+1} t_{2i-1}^{n+1}} - A_{t_{2i-1}^{n+1} t_{2i}^{n+1}} \right) \right| \\ &\leq \sum_{i=1}^{2^n} \left| (\delta_2 A)_{t_{2i-2}^{n+1} t_{2i-1}^{n+1} t_{2i}^{n+1}} \right| \lesssim 2^{-n(\alpha+\beta-1)}\end{aligned}$$

which is summable. Then we obtain that $I_t^n \to I_t$ as $n \to +\infty$ (see again ▸ MG)

If $\alpha = \beta > 1/2$

Theorem

If $f, g \in C^\alpha$, with $\alpha > 1/2$ then there exists a unique pair $(I, R) \in C^\alpha \times C_2^{2\alpha}$ such that

$$I_0 = 0, \qquad I_t - I_s = f_s(g_t - g_s) + R_{st}.$$

In the above situation, we write

$$I_t =: I_{[0,t]}(f, g) =: \int_0^t f \, \mathrm{d}g.$$

Then uniqueness yields the Integration by parts formula

$$I_{[0,t]}(f, g) + I_{[0,t]}(g, f) = f_t g_t - f_0 g_0,$$

since

$$\underbrace{f_t g_t - f_s g_s}_{I_t - I_s} = \underbrace{f_s(g_t - g_s) + g_s(f_t - f_s)}_{A_{st}} + \underbrace{(f_t - f_s)(g_t - g_s)}_{R_{st}}.$$

If $\alpha = \beta \leq 1/2$

However, if $\alpha = \beta \leq 1/2$ then neither existence nor uniqueness.

This problem is revelant for stochastic integration and SDEs:

$$X_t = X_0 + \int_0^t \sigma(X_s)\,\mathrm{d}B_s$$

with $(B_t)_{t \geq 0}$ a standard Brownian motion.

In particular, we can not apply the Sewing Lemma to the germ $A_{st} := f_s(g_t - g_s)$ since $2\alpha \leq 1$ and therefore in general $\delta_2 A \notin C_3^{1+}$.

We need to change the germ A in such a way that $\delta_2 A \in C_3^{1+}$.

Modifying the germ

Note that the result of the integration map is supposed to satisfy

$$I_t - I_s = f_s(g_t - g_s) + R_{st}, \qquad R \in C_2^{2\alpha}.$$

Then we could assume that also f satisfies

$$f_t - f_s = f'_s(g_t - g_s) + R'_{st}, \qquad R' \in C_2^{2\alpha}.$$

If $Y \in C_2$ is such that $(\delta_2 Y)_{sut} = (g_u - g_s)(g_t - g_u)$, setting

$$A_{st} := f_s(g_t - g_s) + f'_s Y_{st},$$

then

$$(\delta_2 A)_{sut} = -\underbrace{(f_u - f_s - f'_s(g_u - g_s))}_{R'_{su}}(g_t - g_u) \in C_3^{3\alpha}.$$

If $1/3 < \alpha \le 1/2$ we are in the setting of the Sewing Lemma.

Rough paths

For $g \in C^\alpha$, we want $Y \in C_2$ such that $(\delta_2 Y)_{sut} = (g_u - g_s)(g_t - g_u)$.

In fact, for $g : [0,T] \to \mathbb{R}$ it is enough to set $Y_{st} := \frac{1}{2}(g_t - g_s)^2$, since $(a+b)^2 - a^2 - b^2 = 2ab$.

This is a natural choice, which moreover shows how much all this is related to generalised Taylor expansions.

However it is not the only possible choice, nor necessarily the most desirable. As we'll see below, Itô integration is not covered by this setting.

In fact, for any such Y we can set $Y' := Y + \delta_1 h$ and Y' still has the desired property.

Note that $Y_{st} = \frac{1}{2}(g_t - g_s)^2$ belongs to $C_2^{2\alpha}$. For reasons which will be clear later, we require this property for all Y.

Rough and controlled paths

Let us summarise: given $\alpha \in]1/3, 1/2]$ and $g \in C^\alpha$, we call a pair $(g, Y) \in C^\alpha \times C_2^{2\alpha}$ a Rough Path if

$$(\delta_2 Y)_{sut} = (g_u - g_s)(g_t - g_u), \qquad 0 \le s \le u \le t \le T.$$

A pair $(f, f') \in C^\alpha \times C^\alpha$ is controlled by g if

$$|f_t - f_s - f'_s(g_t - g_s)| \lesssim |t - s|^{2\alpha}.$$

We denote by $\mathscr{D}_g^{2\alpha}$ the space of paths controlled by g.

Integration of controlled paths

In this setting, we can apply the Sewing Lemma to the germ $A_{st} := f_s(g_t - g_s) + f_s' Y_{st}$ and define the integral $I \in C^\alpha$ such that

$$\delta_1 I = A - \Lambda \delta_2 A, \qquad I_0 = 0.$$

Then the integration map acts (continuously) on controlled paths

$$\mathscr{D}_g^{2\alpha} \ni (f, f') \mapsto (I, f) \in \mathscr{D}_g^{2\alpha}.$$

Brownian motion in $\mathbb{R}$

Let us suppose that $g \equiv B$, a standard Brownian motion in $\mathbb{R}$. Then for all $\alpha < 1/2$, a.s. $B \in C^\alpha$. We fix $\alpha \in]1/3, 1/2]$.

We set $Y_{st} = \frac{1}{2}(B_t - B_s)^2$. For all $\alpha < 1/2$, a.s. $Y \in C_2^\alpha$.

A path controlled by B is $(f, f') \in C^\alpha \times C^\alpha$ such that

$$|f_t - f_s - f'_s(B_t - B_s)| \lesssim |t-s|^{2\alpha}, \qquad 0 \le s \le t \le T.$$

For all such (f, f') there exists a unique $I \in C^\alpha$ such that $I_0 = 0$ and

$$|I_t - I_s - f_s(B_t - B_s) - f'_s Y_{st}| \lesssim |t-s|^{3\alpha}, \qquad 0 \le s \le t \le T.$$

Moreover

$$|I_t - I_s - f_s(B_t - B_s)| \lesssim |t-s|^{2\alpha}, \qquad 0 \le s \le t \le T.$$

If the Stratonovich integral $\int_0^\bullet f_s \circ \mathrm{d}B_s$ is well defined, it is equal to I.

Brownian motion in $\mathbb{R}$

Let us suppose that $g \equiv B$, a standard Brownian motion in $\mathbb{R}$. Then for all $\alpha < 1/2$, a.s. $B \in C^\alpha$. We fix $\alpha \in]1/3, 1/2]$.

We set $Y_{st} = \frac{1}{2}[(B_t - B_s)^2 - (t-s)]$. For all $\alpha < 1/2$, a.s. $Y \in C_2^\alpha$.

A path controlled by B is $(f, f') \in C^\alpha \times C^\alpha$ such that

$$|f_t - f_s - f'_s(B_t - B_s)| \lesssim |t-s|^{2\alpha}, \qquad 0 \le s \le t \le T.$$

For all such (f, f') there exists a unique $I \in C^\alpha$ such that $I_0 = 0$ and

$$|I_t - I_s - f_s(B_t - B_s) - f'_s Y_{st}| \lesssim |t-s|^{3\alpha}, \qquad 0 \le s \le t \le T.$$

Moreover

$$|I_t - I_s - f_s(B_t - B_s)| \lesssim |t-s|^{2\alpha}, \qquad 0 \le s \le t \le T.$$

If the Itô integral $\int_0^\bullet f_s \, \mathrm{d}B_s$ is well defined, it is equal to I.

Multi-dimensional (rough) paths

It is important to extend the above setting to functions $g : [0, T] \to \mathbb{R}^d$.

If $\alpha \in]1/3, 1/2]$ and $g \in C^\alpha$, we call $(g^i, Y^{ij}, 1 \le i, j \le d)$, with $(g^i, Y^{ij}) \in C^\alpha \times C_2^{2\alpha}$ a Rough Path if for all i, j

$$(\delta_2 Y^{ij})_{sut} = (g^i_u - g^i_s)(g^j_t - g^j_u), \qquad 0 \le s \le u \le t \le T.$$

We say that $(f, f'^i) \in C^\alpha \times (C^\alpha)^d$ is controlled by g if

$$|f_t - f_s - \sum_i f'^i_s (g^i_t - g^i_s)| \lesssim |t - s|^{2\alpha}.$$

We denote by $\mathscr{D}_g^{2\alpha}$ the space of paths controlled by g.

In this setting, we can apply the Sewing Lemma to the germ $A^j_{st} := f_s(g^j_t - g^j_s) + \sum_i f'^i_s Y^{ij}_{st}$ and define the integral $I^i \in C^\alpha$ such that

$$\delta_1 I^j = A^j - \Lambda \delta_2 A^j, \qquad I^j_0 = 0.$$

Multi-dimensional (rough) paths

First, this allows to cover SDEs in $\mathbb{R}^d$

$$X_t = X_0 + \int_0^t \sigma(X_s)\,\mathrm{d}B_s, \quad X, B \in C([0,T];\mathbb{R}^d), \ \ \sigma : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d.$$

Furthermore, the situation is more insteresting and complicated, since there is no canonical choice for the off-diagonal terms

$$(\delta_2 Y^{ij})_{sut} = (g^i_u - g^i_s)(g^j_t - g^j_u), \qquad i \neq j.$$

It is always possible to find $Y^{ij} \in C_2$ satisfying this, take e.g. $Y^{ij}_{st} = -g^i_s(g^j_t - g^j_s)$. However in general this choice does not satisfy the analytical requirement $Y^{ij} \in C_2^{2\alpha}$.

Therefore existence of Rough Paths over a path $g : [0,T] \to \mathbb{R}^d$ is not obvious.

Brownian motion in $\mathbb{R}^d$

Let us suppose that $g^i \equiv B^i$, with $B = (B^1, \dots, B^d)$ a standard Brownian motion in $\mathbb{R}^d$. We fix $\alpha \in]1/3, 1/2]$.

We set $Y^{ij}_{st} = \int_s^t (B^i_u - B^i_s) \circ \, \mathrm{d}B^j_u$. For all $\alpha < 1/2$, a.s. $Y \in C_2^{2\alpha}$ (not obvious).

A path controlled by B is $(f, f') \in C^\alpha \times (C^\alpha)^d$ such that

$$|f_t - f_s - \sum_i f'^i_s (B^i_t - B^i_s)| \lesssim |t-s|^{2\alpha}, \qquad 0 \le s \le t \le T.$$

For all such (f, f') there exists a unique $I \in (C^\alpha)^d$ such that $I_0 = 0$ and

$$|I^j_t - I^j_s - f_s(B^j_t - B^j_s) - \sum_i f'^i_s Y^{ij}_{st}| \lesssim |t-s|^{3\alpha}, \qquad 0 \le s \le t \le T.$$

If the Stratonovich integral $\int_0^\bullet f_s \circ \, \mathrm{d}B_s$ is well defined, it is equal to I.

Brownian motion in $\mathbb{R}^d$

Let us suppose that $g^i \equiv B^i$, with $B = (B^1, \dots, B^d)$ a standard Brownian motion in $\mathbb{R}^d$. We fix $\alpha \in]1/3, 1/2]$.

We set $Y^{ij}_{st} = \int_s^t (B^i_u - B^i_s)\,\mathrm{d}B^j_u$. For all $\alpha < 1/2$, a.s. $Y \in C_2^{2\alpha}$ (not obvious).

A path controlled by B is $(f, f') \in C^\alpha \times (C^\alpha)^d$ such that

$$|f_t - f_s - \sum_i f'^i_s (B^i_t - B^i_s)| \lesssim |t-s|^{2\alpha}, \qquad 0 \le s \le t \le T.$$

For all such (f, f') there exists a unique $I \in (C^\alpha)^d$ such that $I_0 = 0$ and

$$|I^j_t - I^j_s - f_s(B^j_t - B^j_s) - \sum_i f'^i_s Y^{ij}_{st}| \lesssim |t-s|^{3\alpha}, \qquad 0 \le s \le t \le T.$$

If the Itô integral $\int_0^\bullet f_s\,\mathrm{d}B_s$ is well defined, it is equal to I.

Remarks

- In the Young situation ($\alpha > 1/2$), f and g play symmetric rôles. The integral is a bilinear functional
- If $\alpha \leq 1/2$, the pair (g, Y) is a non-linear object by the constraint on $\delta_2 Y$.
- In particular, rough paths are non-linear objects. This is where algebra gets into the picture.
- On the other hand, for a fixed rough path, controlled paths form a linear space and the integral is a linear functional.
- The off-diagonal terms $Y^{ij}_{st} = \int_s^t (B^i_u - B^i_s)\, \mathrm{d}B^j_u$, $i \neq j$, are defined using Stochastic calculus. Since $\delta_2 Y^{ij} \in C_3^{1-}$, the Sewing Lemma can not be used to define them.

Remarks

Another fundamental remark:

- the analytical bound in the Sewing Lemma implies that the integral is continuous w.r.t. (f, g, Y).
- This implies that solutions to a Rough Differential Equation are continuous w.r.t. the underlying rough path.
- This was the motivation of Terry Lyons when he introduced Rough Paths in the first place, and it is called the Continuity of the Itô-Lyons map.
- (Hans Föllmer wrote in the '80s a famous note conjecturing this kind of results)
- In the classical theory of stochastic calculus and SDEs, one has in general only measurability of the Itô map.

Lower regularity

If we want to consider a path $g:[0,T]\to\mathbb{R}$ with even lower regularity, say $g\in C^\alpha$ with $\alpha\in\,]1/4,1/3]$, then we have to modify further the germ.

We assume that $(f,f',f'')\in(C^\alpha)^3$ satisfies

$$f_t-f_s=f'_s(g_t-g_s)+f''_s\frac{(g_t-g_s)^2}{2}+R_{st},\qquad R\in C_2^{3\alpha}.$$

Then the germ

$$A_{st}:=f_s(g_t-g_s)+f'_s\frac{(g_t-g_s)^2}{2}+f''_s\frac{(g_t-g_s)^3}{3!}$$

satisfies

$$(\delta_2A)_{sut}=-R_{su}(g_t-g_u)-\underbrace{(f'_t-f'_s-f''_s(g_t-g_s))}_{=:R'_{st}}\frac{(g_t-g_u)^2}{2}.$$

In order to apply the Sewing Lemma, we need that $R'\in C_2^{2\alpha}$.

Lower regularity

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We assume that $(f, f', f'') \in (C^\alpha)^3$ satisfies

$$f_t - f_s = f'_s (g_t - g_s) + f''_s \frac{(g_t - g_s)^2}{2} + R_{st}, \qquad R \in C_2^{3\alpha},$$

$$f'_t - f'_s = f''_s (g_t - g_s) + R'_{st}, \qquad R' \in C_2^{2\alpha}.$$

Then the germ

$$A_{st} := f_s (g_t - g_s) + f'_s \frac{(g_t - g_s)^2}{2} + f''_s \frac{(g_t - g_s)^3}{3!}$$

satisfies (exercise...)

$$(\delta_2 A)_{sut} = -R_{su}(g_t - g_u) - R'_{su} \frac{(g_t - g_u)^2}{2}.$$

If $1/4 < \alpha \leq 1/3$ we are in the setting of the Sewing Lemma.

Compact notations

Let $\alpha \in]0,1[$ and $g \in C^\alpha$.

We set $\mathbb{X}^n_{st} := \frac{1}{n!}(g_t - g_s)^n$, $s,t \in [0,T]$, $n \geq 0$. By Newton's binomial theorem

$$\mathbb{X}^n_{st} = \sum_{k=0}^{n} \mathbb{X}^k_{su}\, \mathbb{X}^{n-k}_{ut}, \qquad s,u,t \in [0,T]$$

(a convolution product...). Note that $\mathbb{X}^n \in C_2^{n\alpha}$ and

$$(\delta_2 \mathbb{X}^n)_{sut} = \sum_{k=1}^{n-1} \mathbb{X}^k_{su}\, \mathbb{X}^{n-k}_{ut}, \qquad s,u,t \in [0,T].$$

Now we define N as the largest integer such that $N\alpha \leq 1$, i.e. $N = \lfloor 1/\alpha \rfloor$.

We say that $Z : [0,T] \to \mathbb{R}^{\{0,\dots,N-1\}}$ is controlled by $\mathbb{X}$ if

$$Z^n_t = \sum_{k=n}^{N-1} Z^k_s\, \mathbb{X}^{k-n}_{st} + R^n_{st}, \qquad n \in \{0,\dots,N-1\},\ R^n \in C_2^{(N-n)\alpha}.$$

Compact notations

Then the germ $$A_{st} := \sum_{k=0}^{N-1} Z_s^k \, \mathbb{X}_{st}^{k+1}$$ satisfies

$$(\delta_2 A)_{sut} = \sum_{k=0}^{N-1} \left[Z_s^k \, (\mathbb{X}_{st}^{k+1} - \mathbb{X}_{su}^{k+1}) - Z_u^k \, \mathbb{X}_{ut}^{k+1} \right]$$

$$= \sum_{k=0}^{N-1} Z_s^k \sum_{i=1}^{k+1} \mathbb{X}_{su}^{k+1-i} \, \mathbb{X}_{ut}^{i} - \sum_{k=0}^{N-1} Z_u^k \, \mathbb{X}_{ut}^{k+1}$$

$$= \sum_{i=0}^{N-1} \mathbb{X}_{ut}^{i+1} \sum_{k=i}^{N-1} Z_s^k \, \mathbb{X}_{su}^{k-i} - \sum_{i=0}^{N-1} Z_u^i \, \mathbb{X}_{ut}^{i+1}$$

$$= \sum_{i=0}^{N-1} \mathbb{X}_{ut}^{i+1} \left[Z_u^i - R_{su}^i \right] - \sum_{i=0}^{N-1} Z_u^i \, \mathbb{X}_{ut}^{i+1}$$

$$= - \sum_{i=0}^{N-1} R_{su}^i \, \mathbb{X}_{ut}^{i+1} \in C_3^{(N-i+i+1)\alpha} \subset C_3^{1+}.$$

Compact notations

We define as above I by $I_0 = 0$ and

$$\delta_1 I = A - \Lambda\delta_2 A, \qquad \bar{R} := -\Lambda\delta_2 A.$$

If we set $\bar{Z} : [0,T] \to \mathbb{R}^{\{0,\dots,N-1\}}$ by

$$\bar{Z}^0_t = I_t, \qquad \bar{Z}^n_t := Z^{n-1}_t, \quad n \in \{1,\dots,N-1\},$$

then $\bar{Z}$ is a controlled path. Indeed

$$\bar{Z}^0_t - \sum_{k=0}^{N-1} \bar{Z}^k_s \, \mathbb{X}^k_{st} = I_t - I_s - \sum_{i=0}^{N-2} Z^i_s \, \mathbb{X}^{i+1}_{st} = [\delta_1 I - A]_{st} + Z^{N-1}_s \, \mathbb{X}^N_{st} \in C^{N\alpha}_2.$$

$$\bar{Z}^n_t = Z^{n-1}_t = \sum_{k=n-1}^{N-1} Z^k_s \, \mathbb{X}^{k-n+1}_{st} + R^n_{st} = \sum_{k=n}^{N-1} \bar{Z}^k_s \, \mathbb{X}^{k-n}_{st} + R^n_{st} + Z^{N-1}_s \mathbb{X}^{N-n}_{st}.$$

Iterated integrals

In four celebrated papers (1954, 1957, 1958, 1971) Kuo-Tsai Chen discovered that the family of iterated integrals of a smooth path in $\mathbb{R}^d$ has a number of algebraic properties.

Let $s \leq t$ and $X : [s,t] \to \mathbb{R}^d$ a smooth path. Set $\mathbb{X}_{st}() := 1$,

$$\begin{aligned}\mathbb{X}_{st}(i_1 \dots i_n) &= \int_s^t \mathbb{X}_{sr}(i_1 \dots i_{n-1})\, \dot X_r^{i_n}\, \mathrm{d}r \\ &= \int_s^t \dot X_{r_n}^{i_n}\, \mathrm{d}r_n \int_s^{r_n} \dot X_{r_{n-1}}^{i_{n-1}}\, \mathrm{d}r_{n-1} \cdots \int_s^{r_2} \dot X_{r_1}^{i_1}\, \mathrm{d}r_1,\end{aligned}$$

with $n \in \mathbb{N}$, $i_k \in \{1, \dots, d\}$.

Then $\mathbb{X}_{st}$ is in the dual V^* of the vector space V spanned by all finite words $\{(a_1 \dots a_n)\}_{n \geq 0}$ with letters in $\{1, \dots, d\}$ (tensor algebra).

Example:

$$\mathbb{X}_{st}(\underbrace{i \dots i}_{n}) = \frac{1}{n!}(X_t^i - X_s^i)^n.$$

Bialgebra

On V we have a bialgebra structure (defined by Frédéric on Monday)

- the shuffle product $\sqcup\!\sqcup : V \otimes V \to V$

$$i\sigma \sqcup\!\sqcup j\tau = i(\sigma \sqcup\!\sqcup j\tau) + j(i\sigma \sqcup\!\sqcup \tau).$$

- the deconcatenation coproduct $\Delta : V \to V \otimes V$

$$\Delta(i_1 \dots i_n) := \sum_{k=0}^{n} (i_1 \dots i_k) \otimes (i_{k+1} \dots i_n)$$

- associativity $\sqcup\!\sqcup(\mathrm{id} \otimes \sqcup\!\sqcup) = \sqcup\!\sqcup(\sqcup\!\sqcup \otimes \mathrm{id})$
- coassociativity $(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta$
- unit $\mathbf{1} : \mathbb{R} \to V$, $\sqcup\!\sqcup(\mathrm{id} \otimes \mathbf{1})(v, r) = \sqcup\!\sqcup(\mathbf{1} \otimes \mathrm{id})(r, v) = rv$
- counit $\mathbf{1}^* : V \to \mathbb{R}$, $(\mathrm{id} \otimes \mathbf{1}^*)\Delta = (\mathbf{1}^* \otimes \mathrm{id})\Delta = \mathrm{id}$
- compatibility $\Delta(a \sqcup\!\sqcup b) = (\Delta a) \sqcup\!\sqcup (\Delta b)$
- grading $V = \oplus_{n \geq 0} V_n$ where V_n is the span of the words with n letters.

Convolution product

Note the recursive formulae $\Delta() = () \otimes ()$,

$$\Delta(\tau i) = (\mathrm{id} \otimes \cdot i)\Delta\tau + \tau i \otimes ().$$

If V has a coproduct, then on V^* we can define the convolution product $\star : V^* \otimes V^* \to V^*$

$$(A \star B)(\tau) := (A \otimes B)\Delta\tau$$

which is associative with unit $\mathbf{1}^*$.

E.g.

$$\langle \mathbb{X}_{su} \star \mathbb{X}_{ut}, \tau \rangle = \langle \mathbb{X}_{su} \otimes \mathbb{X}_{ut}, \Delta\tau \rangle, \qquad \forall \tau \in V.$$

Important remark: $\star$ is commutative if and only if Δ is cocommutative.

(Deconcatenation is not cocommutative)

Hopf Algebra

If V is a bialgebra and we have a linear map $\mathcal{A} : V \to V$ (antipode) such that for all $\tau \in V$

$$\sqcup\!\sqcup(\mathcal{A} \otimes \mathrm{id})\Delta\tau = \sqcup\!\sqcup(\mathrm{id} \otimes \mathcal{A})\Delta\tau = \mathbf{1} \circ \mathbf{1}^*(\tau)$$

then V is called a Hopf Algebra. In our case:

$$\mathcal{A}(i_1 \dots i_n) = (-1)^n (i_n \dots i_1).$$

Let $G \subset V^*$ the space of characters (multiplicative functionals):

$$g \in V^*, \qquad g(a \sqcup\!\sqcup b) = g(a)\, g(b), \qquad \forall\, a, b \in V.$$

If V is a Hopf Algebra then G is a group for the convolution product

$$(g_1 \star g_2)(\tau) := (g_1 \otimes g_2)\Delta\tau$$

with inverse $g^{-1} = g \circ \mathcal{A}$ and identity $\mathbf{1}^*$.

Concatenation

If $u \in [s,t]$ then $X_{[s,t]} := (X_r, r \in [s,t])$ is the concatenation of $X_{[s,u]}$ and $X_{[u,t]}$. We write

$$X_{[s,t]} = X_{[s,u]} * X_{[u,t]}.$$

Setting $r_{n+1} := t$, $r_0 := s$, we have

$$\begin{aligned} &\mathbb{X}_{st}(i_1 \dots i_n) = \\ &= \sum_{k=0}^{n} \int_s^t \dot{X}_{r_n}^{i_n}\, \mathrm{d}r_n \int_s^{r_n} \dot{X}_{r_{n-1}}^{i_{n-1}}\, \mathrm{d}r_{n-1} \cdots \int_s^{r_2} \dot{X}_{r_1}^{i_1}\, \mathrm{d}r_1 \, \mathbb{1}_{(r_k \le u < r_{k+1})} \\ &= \sum_{k=0}^{n} \mathbb{X}_{su}(i_1 \dots i_k)\, \mathbb{X}_{ut}(i_{k+1} \dots i_n). \end{aligned}$$

Namely $\mathbb{X}_{st} = \mathbb{X}_{su} \star \mathbb{X}_{ut}$,

$$\langle \mathbb{X}_{st}, \tau \rangle = \langle \mathbb{X}_{su} \otimes \mathbb{X}_{ut}, \Delta\tau \rangle = \langle \mathbb{X}_{su} \star \mathbb{X}_{ut}, \tau \rangle, \qquad \forall \tau \in V.$$

Shuffle

Note now that

$$\mathbb{1}_{(s<r_1<\cdots<r_n<t)}\,\mathbb{1}_{(s<r_{n+1}<\cdots<r_{n+m}<t)} = \sum_{\sigma\in\mathrm{Sh(n,m)}} \mathbb{1}_{(s<r_{\sigma(1)}<\cdots<r_{\sigma(n+m)}<t)}$$

where $\mathrm{Sh(n,m)}$ is the set of all $\sigma \in S_{n+m}$ such that

$$\sigma^{-1}(1) < \sigma^{-1}(2) < \ldots < \sigma^{-1}(n),$$

$$\sigma^{-1}(n+1) < \sigma^{-1}(n+2) \ldots < \sigma^{-1}(n+m).$$

This yields the multiplicativity w.r.t. the shuffle product

$$\langle \mathbb{X}_{st}, \tau_1 \rangle \langle \mathbb{X}_{st}, \tau_2 \rangle = \langle \mathbb{X}_{st}, \tau_1 \sqcup\!\sqcup\, \tau_2 \rangle$$

$$(i_1 \ldots i_n) \sqcup\!\sqcup\, (i_{n+1} \ldots i_{n+m}) = \sum_{\sigma\in\mathrm{Sh(n,m)}} (i_{\sigma(1)} \ldots i_{\sigma(n+m)}).$$

Geometric rough paths

Chen proved that $\mathbb{X}$ is a V^*-valued function with the following properties for all $s \leq u \leq t$:

- $\mathbb{X}_{st}(\tau) = (\mathbb{X}_{su} \otimes \mathbb{X}_{ut})\Delta\tau, \forall \tau \in V$, i.e. $\mathbb{X}_{su} \star \mathbb{X}_{ut} = \mathbb{X}_{st}$.
- $\mathbb{X}_{st}(\tau_1 \sqcup\!\sqcup \tau_2) = \mathbb{X}_{st}(\tau_1)\mathbb{X}_{st}(\tau_2)$.

(Notations from [Hairer-Kelly 2013]).

Therefore $\mathbb{X}$ is a flow of characters.

Terry Lyons defined [1998] a (weak) geometric rough path of regularity $\alpha > 0$ as a V^*-valued function $\mathbb{X}$ satisfying the above properties plus some control on the modulus of continuity

- $\sup_{s \neq t}[|\mathbb{X}_{st}(i_1 \ldots i_n)|/|t-s|^{n\alpha}] < +\infty$, for all $(i_1 \ldots i_n) \in V$.

Remarks:

- Smooth paths are dense.
- $\mathbb{X}_{st}(i) = X^i_t - X^i_s$ for some $X^i \in C^\alpha$, since i is primitive in V.

Rough integration and differential equations

Terry Lyons proved that this setting allows to give a deterministic theory of integration w.r.t. $\mathrm{d}X$ and to solve differential equations

$$\mathrm{d}Y = \alpha(Y)\,\mathrm{d}X,$$

obtaining continuity of the Itô-Lyons map $\mathbb{X} \mapsto Y$ and even $\mathbb{X} \mapsto \mathbb{Y}$, although the map $X \mapsto Y$ is in general only measurable.

This result includes Brownian integration, both in the sense of Itô and Stratonovich (although the Itô rough path is not geometric), but not more general rough paths.

Note that setting $\mathbb{X}_t := \mathbb{X}_{0t}$, we have

$$\mathbb{X}_{st} = \mathbb{X}_s^{-1} \star \mathbb{X}_t = (\mathbb{X}_s \circ \mathcal{A}) \star \mathbb{X}_t$$

where $\mathcal{A}$ is the antipode.

The Stratonovich Rough Path is geometric

Let $(B^i_t)_{i\geq 1, t\geq 0}$ be independent Brownian Motions.

We set $\mathbb{X}_{st}() := 1$ and for $n \geq 1$

$$\mathbb{X}_{st}(i_1 \ldots i_n) := \int_s^t \mathbb{X}_{sr}(i_1 \ldots i_{n-1}) \circ \, \mathrm{d}B^{i_n}_r.$$

We claim that this defines a.s. a geometric rough path.

The Chen relation

A recurrence proof: let us set $\tau = (i_1, \dots, i_{n-1})$ and $\tau_i = (i_1, \dots, i_{n-1}, i)$. Then

$$\begin{aligned}
\langle \mathbb{X}_{st}, \tau_i \rangle &= \int_s^t (\mathbb{X}_{sr}\tau) \circ \mathrm{d}B^i_r \\
&= \int_s^u (\mathbb{X}_{sr}\tau) \circ \mathrm{d}B^i_r + \int_u^t (\mathbb{X}_{sr}\tau) \circ \mathrm{d}B^i_r \\
&= \langle \mathbb{X}_{su}, \tau_i \rangle + \int_u^t \langle \mathbb{X}_{su} \otimes \mathbb{X}_{ur}, \Delta\tau \rangle \circ \mathrm{d}B^i_r \\
&= \langle \mathbb{X}_{su}, \tau_i \rangle + \langle \mathbb{X}_{su} \otimes \int_u^t \mathbb{X}_{ur} \circ \mathrm{d}B^i_r, \Delta\tau \rangle \\
&= \langle \mathbb{X}_{su} \otimes \mathbb{X}_{ut}, \tau_i \otimes 1 + (\mathrm{id} \otimes \cdot i)\Delta\tau \rangle \\
&= \langle \mathbb{X}_{su} \otimes \mathbb{X}_{ut}, \Delta\tau_i \rangle.
\end{aligned}$$

Multiplicativity w.r.t. the shuffle

Recall that for M, N two continuous semimartingales, the Stratonovich integral has the property

$$M_t N_t - M_s N_s = \int_s^t M_r \circ \mathrm{d}N_r + \int_s^t N_r \circ \mathrm{d}M_r$$

(integration by parts formula).

This implies

$$\mathbb{X}_{st}(i)\, \mathbb{X}_{st}(j) = \int_s^t \mathbb{X}_{sr}(i) \circ \mathrm{d}B_r^j + \int_s^t \mathbb{X}_{sr}(j) \circ \mathrm{d}B_r^i$$
$$= \mathbb{X}_{st}(ij) + \mathbb{X}_{st}(ji) = \mathbb{X}_{st}(i \sqcup\!\sqcup j).$$

Multiplicativity w.r.t. the shuffle

Let $M_t := \mathbb{X}_{st}(\tau i)$, $N_t := \mathbb{X}_{st}(\sigma j)$, $t \geq s$. Then

$$\mathbb{X}_{st}(\tau i)\,\mathbb{X}_{st}(\sigma j) = M_t N_t = \int_s^t M_r \circ \, \mathrm{d}N_r + \int_s^t N_r \circ \, \mathrm{d}M_r =$$
$$= \int_s^t \mathbb{X}_{sr}\tau \, \mathbb{X}_{sr}(\sigma j) \circ \, \mathrm{d}B^i_r + \int_s^t \mathbb{X}_{sr}\sigma \, \mathbb{X}_{sr}(\tau i) \circ \, \mathrm{d}B^j_r$$
$$= \int_s^t \mathbb{X}_{sr}(\tau \sqcup\!\sqcup \sigma j) \circ \, \mathrm{d}B^i_r + \int_s^t \mathbb{X}_{sr}(\tau i \sqcup\!\sqcup \sigma) \circ \, \mathrm{d}B^j_r$$
$$= \mathbb{X}_{st}\left((\tau \sqcup\!\sqcup \sigma j)i + (\tau i \sqcup\!\sqcup \sigma)j\right) = \mathbb{X}_{st}(\tau i \sqcup\!\sqcup \sigma j).$$

The extension Theorem

Theorem (T. Lyons)

Given a (geometric) rough path $\mathbb{X}$ *of regularity* $\alpha > 0$, *the values* $(\mathbb{X}\tau, \tau \in V_m, m > N)$ *are uniquely determined by the values of* $(\mathbb{X}\tau, \tau \in V_m, m \leq N)$, *where* $N := \lfloor 1/\alpha \rfloor$.

Proof.

We have for all $\tau \in V_m$

$$(\delta_2 \mathbb{X}\tau)_{sut} = (\mathbb{X}_{su} \otimes \mathbb{X}_{ut})\Delta'\tau$$

where $\Delta'\tau := \Delta\tau - () \otimes \tau - \tau \otimes ()$ is the reduced coproduct. We conclude by recurrence on the number of letters and by the Sewing Lemma since $(\delta_2 \mathbb{X}\tau) \in C_3^{m\alpha}$. □

Controlled Paths

Given a geometric rough path $\mathbb{X}$ of regularity $\alpha > 0$, we say that $Z : [0, T] \to V_{N-1}$, with $N := \lfloor 1/\alpha \rfloor$, is a controlled path if for all words τ, σ

$$Z_t^\tau = \sum_{|\sigma| \le N-1} Z_s^\sigma \, (\mathbb{X}_{st} \otimes \tau^*) \Delta\sigma + R_{st}^\tau, \qquad R^\tau \in C_2^{(N-|\tau|)\alpha},$$

where $\tau^* : V \to \mathbb{R}$ is the linear functional such that $\tau^*(\sigma) = \mathbb{1}_{(\tau=\sigma)}$ and $|\sigma|$ is the number of letters in σ.

When the alphabet has a single letter, the condition is:
We say that $Z : [0, T] \to \mathbb{R}^{\{0,\dots,N-1\}}$ is controlled by $\mathbb{X}$ if

$$Z_t^n = \sum_{k=n}^{N-1} Z_s^k \, \mathbb{X}_{st}^{k-n} + R_{st}^n, \qquad n \in \{0, \dots, N-1\}, \; R^n \in C_2^{(N-n)\alpha}.$$

Rough Integration

Theorem

If Z is a controlled path then for each letter i the germ

$$A^i_{st} := \sum_{|\sigma| \le N-1} Z^\sigma_s \, \mathbb{X}^{\sigma i}_{st}$$

satisfies $\delta_2 A \in C_3^{(N+1)\alpha}$. Then by the Sewing Lemma the rough integral

$$\int_0^\bullet Z \, \mathrm{d}X^i$$

is well defined where $X^i_t - X^i_s = \mathbb{X}_{st}(i)$.

The Itô Rough Path is not geometric

Let $(B^i_t)_{i\geq 1, t\geq 0}$ be independent Brownian Motions.

We set $\mathbb{X}_{st}() := 1$ and

$$\mathbb{X}_{st}(i_1 \dots i_n) := \int_s^t \mathbb{X}_{sr}(i_1 \dots i_{n-1})\, \mathrm{d}B^{i_n}_r.$$

E.g. $i \sqcup\!\sqcup\, i = 2ii$,

$$\mathbb{X}_{st}(i \sqcup\!\sqcup\, i) = 2\int_s^t (B^i_r - B^i_s)\, \mathrm{d}B^i_r = (B^i_t - B^i_s)^2 - (t-s)$$

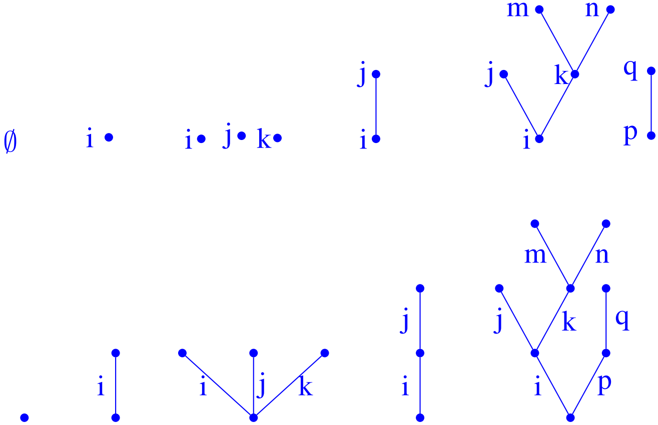
$$\mathbb{X}_{st}(i) = B^i_t - B^i_s \implies \mathbb{X}_{st}(i \sqcup\!\sqcup\, i) \neq \mathbb{X}_{st}(i)\mathbb{X}_{st}(i).$$

However we do have $\mathbb{X}_{st} = \mathbb{X}_{su} \star \mathbb{X}_{ut}$: setting $r_0 := t$, $r_{n+1} := s$

$$\begin{aligned}
&\mathbb{X}_{st}(i_1 \ldots i_n) = \\
&= \sum_{k=0}^{n} \int_s^t \mathrm{d}B^{i_n}_{r_n} \int_s^{r_n} \mathrm{d}B^{i_{n-1}}_{r_{n-1}} \cdots \int_s^{r_2} \mathrm{d}B^{i_1}_{r_1} \, \mathbb{1}_{(r_k \le u < r_{k+1})} \\
&= \sum_{k=0}^{n} \mathbb{X}_{su}(i_1 \ldots i_k) \, \mathbb{X}_{ut}(i_{k+1} \ldots i_n).
\end{aligned}$$

How can we describe the Itô Rough Path?

Two equivalent settings



The Connes-Kreimer Hopf algebra

We consider the space $\mathcal{H}$ of rooted trees, with edges decorated by letters of the alphabet $\{1, \ldots, d\}$. The identity is $\bullet$, the product is the identification of the roots, and the coproduct is

$$\Delta\tau = \sum_{\sigma \subseteq \tau} (\tau/\sigma) \otimes \sigma$$

where σ varies among all subtrees of τ with the same root as τ.

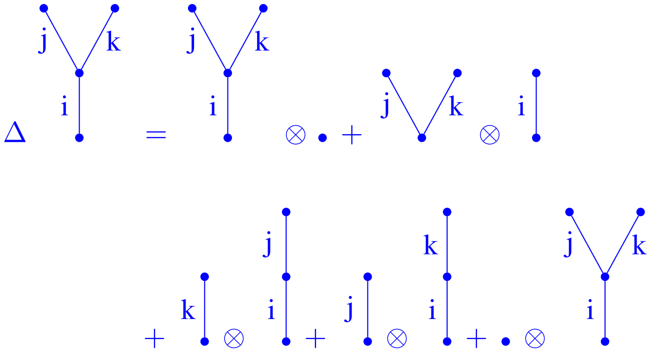
This is a bialgebra and a Hopf algebra.

The previous bialgebra V is canonically embedded in $\mathcal{H}$: a word $(i_1 \cdots i_n)$ is interpreted as a linear tree with n edges, the first (at the root) decorated with i_n, the next with i_{n-1} and so on.

The coproduct of $\mathcal{H}$ extends that of V, the product does not.

This Hopf algebra was already famous in numerical analysis (!): Butcher (1972) and Hairer-Wanner (1974).

An example



(different but isomorphic representation w.r.t. that common in algebra, see Kurusch' lectures).

A recursive formula

$\mathcal{H}$ has a recursive structure: all elements of $\mathcal{H}$ are obtained from $\bullet$ with a finite number of products and of applications of the operators

$$\tau \to [\tau]_i$$

where we add to the root of τ a new edge with decoration i and we move the root to the new node.

The coproduct Δ has the recursive construction

$$\Delta\bullet = \bullet \otimes \bullet, \qquad \Delta(\tau_1 \cdots \tau_n) = (\Delta\tau_1) \cdots (\Delta\tau_n)$$

$$\Delta[\tau]_i = [\tau]_i \otimes \bullet + (\mathrm{id} \otimes [\cdot]_i)\Delta\tau.$$

(A non-cocommutative coproduct)

$\mathcal{H}$ is graded by the number of edges.

Chen and Kreimer

In 1998, Dirk Kreimer gives an extension of Chen's result.

He extends the iterated integrals to functionals of decorated trees in $\mathcal{H}$:

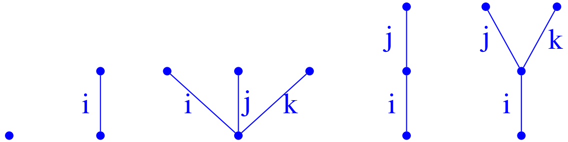
- $\langle \mathbb{X}_{st}, \bullet \rangle = 1$
- $\langle \mathbb{X}_{st}, \tau_1 \cdots \tau_n \rangle = \langle \mathbb{X}_{st}, \tau_1 \rangle \cdots \langle \mathbb{X}_{st}, \tau_n \rangle$
- $$\langle \mathbb{X}_{st}, [\tau]_i \rangle = \int_s^t (\mathbb{X}_{su}\tau)\, \dot{X}_u^i \, \mathrm{d}u$$

and shows that $\mathbb{X}$ is a $\mathcal{H}^*$-valued function with the following properties for all $s \le u \le t$:

- $\mathbb{X}_{st}(\tau) = (\mathbb{X}_{su} \otimes \mathbb{X}_{ut})\Delta\tau$, $\forall \tau \in \mathcal{H}$, i.e. $\mathbb{X}_{su} \star \mathbb{X}_{ut} = \mathbb{X}_{st}$
- $\mathbb{X}_{st}(\tau_1\tau_2) = \mathbb{X}_{st}(\tau_1)\mathbb{X}_{st}(\tau_2)$.

Examples

$\tau =$



- $\mathbb{X}_{st}(\tau) = 1$
- $\mathbb{X}_{st}(\tau) = X^i_t - X^i_s = \int_s^t \dot{X}^i_r \, \mathrm{d}r$
- $\mathbb{X}_{st}(\tau) = (X^i_t - X^i_s)(X^j_t - X^j_s)(X^k_t - X^k_s)$
- $\mathbb{X}_{st}(\tau) = \int_s^t (X^j_r - X^j_s) \, \dot{X}^i_r \, \mathrm{d}r$
- $\mathbb{X}_{st}(\tau) = \int_s^t (X^j_r - X^j_s)(X^k_r - X^k_s) \, \dot{X}^i_r \, \mathrm{d}r$

A recursive proof of Chen's relation

$$
\begin{aligned}
\langle \mathbb{X}_{st}, [\tau]_i \rangle &= \int_s^t (\mathbb{X}_{sr}\tau)\, \dot{X}^i_r \,\mathrm{d}r \\
&= \int_s^u (\mathbb{X}_{sr}\tau)\, \dot{X}^i_r \,\mathrm{d}r + \int_u^t (\mathbb{X}_{sr}\tau)\, \dot{X}^i_r \,\mathrm{d}r \\
&= \langle \mathbb{X}_{su}, [\tau]_i \rangle + \int_u^t \langle \mathbb{X}_{su} \otimes \mathbb{X}_{ur}, \Delta\tau \rangle \dot{X}^i_r \,\mathrm{d}r \\
&= \langle \mathbb{X}_{su}, [\tau]_i \rangle + \langle \mathbb{X}_{su} \otimes \int_u^t \mathbb{X}_{ur}\, \dot{X}^i_r \,\mathrm{d}r, \Delta\tau \rangle \\
&= \langle \mathbb{X}_{su}, [\tau]_i \rangle + \langle \mathbb{X}_{su} \otimes \mathbb{X}_{ut}[\cdot]_i, \Delta\tau \rangle \\
&= \langle \mathbb{X}_{su} \otimes \mathbb{X}_{ut}, [\tau]_i \otimes 1 + (\mathrm{id} \otimes [\cdot]_i)\Delta\tau \rangle \\
&= \langle \mathbb{X}_{su} \otimes \mathbb{X}_{ut}, \Delta[\tau]_i \rangle.
\end{aligned}
$$

Branched rough paths

In 2006 Massimiliano defines a branched rough path of regularity $\alpha > 0$ as a function $\mathbb{X} : [0,T]^2 \to \mathcal{H}^*$ s.t.

- $\langle \mathbb{X}_{st}, \tau \rangle = \langle \mathbb{X}_{su} \otimes \mathbb{X}_{ut}, \Delta\tau \rangle, \quad \forall \tau \in \mathcal{H}$.
- $\langle \mathbb{X}_{st}, \tau_1\tau_2 \rangle = \langle \mathbb{X}_{st}, \tau_1 \rangle \langle \mathbb{X}_{st}, \tau_2 \rangle$.
- $\sup_{s \neq t} [|\langle \mathbb{X}_{st}, \tau \rangle| / |t-s|^{\alpha|\tau|}] < +\infty$, for all $\tau \in \mathcal{H}$, where $|\tau|$ is the number of edges of τ.

Notations and presentation follow [Hairer-Kelly 2013].

Massimiliano also extends the analytical theory of rough SDEs to the branched case, in particular the notion of controlled paths.

Since $[\bullet]_i$ is primitive, we have $\mathbb{X}_{st}([\bullet]_i) = X^i_t - X^i_s$ with $X^i \in C^\alpha$.

Itô as a Branched Rough Path

$$\begin{aligned}
\langle \mathbb{X}_{st}, [\tau]_i \rangle &= \int_s^t (\mathbb{X}_{sr}\tau)\, \mathrm{d}B^i_r \\
&= \int_s^u (\mathbb{X}_{sr}\tau)\, \mathrm{d}B^i_r + \int_u^t (\mathbb{X}_{sr}\tau)\, \mathrm{d}B^i_r \\
&= \langle \mathbb{X}_{su}, [\tau]_i \rangle + \int_u^t \langle \mathbb{X}_{su} \otimes \mathbb{X}_{ur}, \Delta\tau \rangle \, \mathrm{d}B^i_r \\
&= \langle \mathbb{X}_{su}, [\tau]_i \rangle + \langle \mathbb{X}_{su} \otimes \int_u^t \mathbb{X}_{ur}\, \mathrm{d}B^i_r, \Delta\tau \rangle \\
&= \langle \mathbb{X}_{su}, [\tau]_i \rangle + \langle \mathbb{X}_{su} \otimes \mathbb{X}_{ut}[\cdot]_i, \Delta\tau \rangle \\
&= \langle \mathbb{X}_{su} \otimes \mathbb{X}_{ut}, [\tau]_i \otimes 1 + (\mathrm{id} \otimes [\cdot]_i)\Delta\tau \rangle \\
&= \langle \mathbb{X}_{su} \otimes \mathbb{X}_{ut}, \Delta[\tau]_i \rangle.
\end{aligned}$$

Itô as a Branched Rough Path

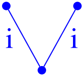
Let us recall that the Itô Branched Rough Path is not geometric, since

$$\mathbb{X}_{st}(i \sqcup\!\sqcup i) = (B^i_t - B^i_s)^2 - (t-s) \neq (B^i_t - B^i_s)^2 = \mathbb{X}_{st}(i)\mathbb{X}_{st}(i).$$

Note that now $i \sqcup\!\sqcup i = 2\tau$ with τ equal to

which is not a product in $\mathcal{H}$. On the other hand,

$\sigma =$ 

$\Longrightarrow \quad \sigma\sigma =$ 

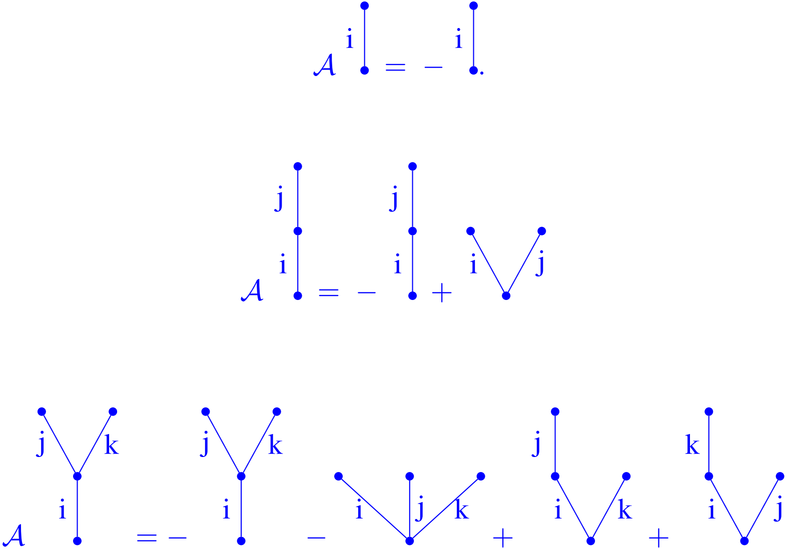
.

Note that setting $\mathbb{X}_t := \mathbb{X}_{0t}$, we have

$$\mathbb{X}_{st} = \mathbb{X}_s^{-1} \star \mathbb{X}_t = (\mathbb{X}_s \circ \mathcal{A}) \star \mathbb{X}_t$$

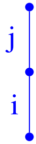
where $\mathcal{A}$ is the antipode in $\mathcal{H}$.

The antipode



Exercise

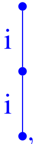
Let $\mathbb{X}$ be the Itô Brownian rough path, with $1/3 \leq \gamma < 1/2$. Then for

$\tau =$ 

$$\begin{aligned} \mathbb{X}_{st}\tau &= \int_s^t (B_r^j - B_s^j)\,\mathrm{d}B_r^i \\ &= \int_0^t B_r^j\,\mathrm{d}B_r^i - \int_0^s B_r^j\,\mathrm{d}B_r^i + B_s^i B_s^j - B_t^i B_s^j \\ &= (\mathbb{X}_s \circ \mathcal{A}) \star \mathbb{X}_t\,\tau. \end{aligned}$$

Let $\mathbb{X}$ be the Itô Brownian rough path, with $1/3 \leq \gamma < 1/2$.

For $s \leq t$, what is $\mathbb{X}_{ts}$? For instance, if



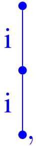
$$\tau := \quad , \qquad \mathbb{X}_{st}\tau = \int_s^t (B_r^i - B_s^i)\,\mathrm{d}B_r^i = \frac{(B_t^i - B_s^i)^2 - (t-s)}{2}.$$

What is $\mathbb{X}_{ts}\tau$?

Question

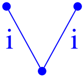
Let $\mathbb{X}$ be the Itô Brownian rough path, with $1/3 \leq \gamma < 1/2$.

For $s \leq t$, what is $\mathbb{X}_{ts}$? For instance, if

$\tau :=$ 

$$\mathbb{X}_{st}\tau = \int_s^t (B^i_r - B^i_s)\, \mathrm{d}B^i_r = \frac{(B^i_t - B^i_s)^2 - (t-s)}{2}.$$

What is $\mathbb{X}_{ts}\tau$? Answer: by the Chen relation, $\mathbb{X}_{ts} = \mathbb{X}_{st} \circ \mathcal{A}$. Example:

$\mathbb{X}_{ts}\tau \;=\; -\mathbb{X}_{st}\tau \;+\; \mathbb{X}_{st}$ 

$$= -\frac{(B^i_t - B^i_s)^2 - (t-s)}{2} + (B^i_t - B^i_s)^2 = \frac{(B^i_t - B^i_s)^2 + (t-s)}{2}.$$

Theorem

Given a branched rough path $\mathbb{X}$ *of regularity* $\alpha > 0$*, the values* $(\mathbb{X}\tau, \tau \in \mathcal{H}_m, m > N)$ *are uniquely determined by the values of* $(\mathbb{X}\tau, \tau \in \mathcal{H}_m, m \leq N)$*, where* $N := \lfloor 1/\alpha \rfloor$.

Proof.

We have for all $\tau \in \mathcal{H}_m$

$$(\delta_2 \mathbb{X}\tau)_{sut} = (\mathbb{X}_{su} \otimes \mathbb{X}_{ut})\Delta'\tau$$

where $\Delta'\tau := \Delta\tau - \bullet \otimes \tau - \tau \otimes \bullet$ is the reduced coproduct. We conclude by recurrence on the number of edges and by the Sewing Lemma since $(\delta_2 \mathbb{X}\tau) \in C_3^{m\alpha}$. □

Controlled Paths

Given a branched rough path $\mathbb{X}$ of regularity $\alpha > 0$, we say that $Z : [0,T] \to \mathcal{H}_{N-1}$, with $N := \lfloor 1/\alpha \rfloor$, is a controlled path if for all trees τ, σ

$$Z_t^\tau = \sum_{|\sigma| \le N-1} Z_s^\sigma \, (\mathbb{X}_{st} \otimes \tau^*)\Delta\sigma + R_{st}^\tau, \qquad R^\tau \in C_2^{(N-|\tau|)\alpha},$$

where $\tau^* : \mathcal{H} \to \mathbb{R}$ is the linear functional such that $\tau^*(\sigma) = \mathbb{1}_{(\tau=\sigma)}$ and $|\sigma|$ is the number of edges in σ.

When the alphabet has a single letter, the condition is:
We say that $Z : [0,T] \to \mathbb{R}^{\{0,\dots,N-1\}}$ is controlled by $\mathbb{X}$ if

$$Z_t^n = \sum_{k=n}^{N-1} Z_s^k \, \mathbb{X}_{st}^{k-n} + R_{st}^n, \qquad n \in \{0,\dots,N-1\}, \; R^n \in C_2^{(N-n)\alpha}.$$

Theorem

If Z is a controlled path then for each letter i the germ

$$A^i_{st} := \sum_{|\sigma|\leq N-1} Z^\sigma_s \, \mathbb{X}^{[\sigma]_i}_{st}$$

satisfies $\delta_2 A \in C_3^{(N+1)\alpha}$. Then by the Sewing Lemma the rough integral

$$\int_0^\bullet Z \, \mathrm{d}X^i$$

is well defined where $X^i_t - X^i_s = \mathbb{X}_{st}([\bullet]_i)$.

Rough Integration

Theorem

If Z is a controlled path then for each letter i the germ

$$A^i_{st} := \sum_{|\sigma| \le N-1} Z^\sigma_s \, \mathbb{X}^{[\sigma]_i}_{st}$$

satisfies $\delta_2 A \in C_3^{(N+1)\alpha}$. Then by the Sewing Lemma the rough integral

$$\int_0^\bullet Z \, \mathrm{d}X^i$$

is well defined where $X^i_t - X^i_s = \mathbb{X}_{st}([\bullet]_i)$.

For further readings on Rough Paths, see the books by Peter Friz.

If you want one paper to read on what I discussed until now, then I recommend [Hairer-Kelly, AIHP15].

Three papers

- Martin Hairer (2014),
  A theory of regularity structures, Inventiones.
- Yvain Bruned, M.H., L.Z. (2016),
  Algebraic renormalisation of regularity structures, arXiv.
- Ajay Chandra, M.H. (2016),
  An analytic BPHZ theorem for regulariy structures, arXiv.

This trio of papers "gives a completely automatic black box for local existence and uniqueness theorems for a wide class of SPDEs".

Singular stochastic PDEs

Let ξ be a space time white noise

$$\text{(KPZ)} \qquad \partial_t u = \Delta u + (\partial_x u)^2 + \xi, \quad x \in \mathbb{R},$$

$$\text{(PAM)} \qquad \partial_t u = \Delta u + u\,\xi, \quad x \in \mathbb{R}^2,$$

$$(\Phi^4_3) \qquad \partial_t u = \Delta u - u^3 + \xi, \quad x \in \mathbb{R}^3.$$

Even for polynomial non-linearities, we do not know how to properly define products of (random) distributions.

Note that if $T \in \mathcal{S}'(\mathbb{R}^d)$ and $\psi \in \mathcal{S}(\mathbb{R}^d)$, then we can define canonically the product $\psi T = T\psi \in \mathcal{S}'(\mathbb{R}^d)$ by

$$\psi T(\varphi) = T\psi(\varphi) := T(\psi\varphi), \qquad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

Similar problem with stochastic integrals, as we have seen.

Wong-Zakai

Let us consider the ODE in $\mathbb{R}^d$

$$\dot{x_\varepsilon} = b(x_\varepsilon) + f(x_\varepsilon)\,\dot{B}_\varepsilon \tag{1}$$

where B_ε is a smooth approximation of a BM B. Then it is well known that $x_\varepsilon \to x$ solution to the Stratonovich SDE

$$\mathrm{d}x = b(x)\,\mathrm{d}t + f(x) \circ \,\mathrm{d}B.$$

In order to obtain the Itô SDE in the limit, one has to define rather

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}_\varepsilon = b(\hat{x}_\varepsilon) - \frac{1}{2}Df(\hat{x}_\varepsilon)f(\hat{x}_\varepsilon) + f(\hat{x}_\varepsilon)\,\dot{B}_\varepsilon \tag{2}$$

and in this case $\hat{x}_\varepsilon \to \hat{x}$ solution to

$$\mathrm{d}\hat{x} = b(\hat{x})\,\mathrm{d}t + f(\hat{x})\,\mathrm{d}B.$$

Now, (2) is a renormalisation of (1).

Regularisation

Let $\xi_\varepsilon = \rho_\varepsilon * \xi$ a regularisation of ξ and let u_ε solve

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + F(u_\varepsilon, \nabla u_\varepsilon, \xi_\varepsilon).$$

What happens as $\varepsilon \to 0$?

We need a topology such that

- the map $\xi_\varepsilon \mapsto u_\varepsilon$ is continuous
- $\xi_\varepsilon \to \xi$ as $\varepsilon \to 0$.

For classical negative Sobolev spaces the first point fails.

For classical positive Sobolev spaces the second point fails.

The theory of regularity structures (RS) gives a framework to solve this problem.

The Solution Map on models

Martin's theory gives

- a space of Models $(\mathcal{M}, \mathrm{d})$ (analog of the space of Rough Paths)
- a canonical lift of every smooth ξ_ε to a model $\mathbb{X}^\varepsilon \in \mathcal{M}$
- a continuous function $\Phi : \mathcal{M} \to \mathcal{S}'(\mathbb{R}^d)$ such that $u_\varepsilon = \Phi(\mathbb{X}^\varepsilon)$ solves the regularised equation

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + F(u_\varepsilon, \nabla u_\varepsilon, \xi_\varepsilon).$$

The model $\mathbb{X}^\varepsilon \in \mathcal{M}$ contains a finite number of relevant explicit products (analogous to the necessary finitely many iterated integrals)

$$\text{e.g.} \qquad \xi_\varepsilon(G * \xi_\varepsilon)$$

(with G the heat kernel). These products can be ill-defined in the limit $\varepsilon \to 0$:

$$\mathbb{E}[\xi_\varepsilon(G * \xi_\varepsilon)] = \rho_\varepsilon * G * \rho_\varepsilon(0) \to G(0) = +\infty.$$

Therefore in general $\mathbb{X}^\varepsilon$ does not converge in $(\mathcal{M}, \mathrm{d})$ as $\varepsilon \to 0$.

Renormalised products

The theory identifies a class of equations, called subcritical, for which it is enough to modify a finite number of products in order to obtain a convergent lift $\hat{\mathbb{X}}^\varepsilon \in \mathcal{M}$ of ξ_ε. For instance

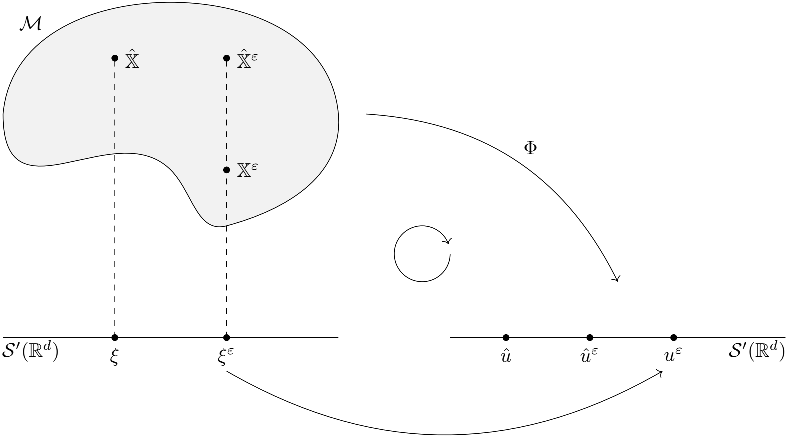
$$\xi_\varepsilon(G * \xi_\varepsilon) \to \xi_\varepsilon(G * \xi_\varepsilon) - \mathbb{E}[\xi_\varepsilon(G * \xi_\varepsilon)].$$

The model $\hat{\mathbb{X}}^\varepsilon \in \mathcal{M}$ contains all these modified (renormalised) products.

Convergence in $(\mathcal{M}, \mathrm{d})$ means (simplifying a lot) convergence of all these objects as distributions.

Then we define the renormalised solution by $\hat{u}_\varepsilon := \Phi(\hat{\mathbb{X}}^\varepsilon)$.

An image


$\mathcal{M}$
$\hat{\mathbb{X}}$
$\hat{\mathbb{X}}^\varepsilon$
$\mathbb{X}^\varepsilon$
Φ
$\mathcal{S}'(\mathbb{R}^d)$
ξ
ξ^ε
$\hat{u}$
$\hat{u}^\varepsilon$
u^ε
$\mathcal{S}'(\mathbb{R}^d)$

The general procedure

One can summarize the procedure into three steps:

- Analytic step Construction of the space of models $(\mathcal{M}, \mathrm{d})$ and continuity of the solution map $\Phi : \mathcal{M} \to \mathcal{S}'(\mathbb{R}^d)$, [MH14]
- Algebraic step Renormalisation of the canonical model $\mathbb{X}^\varepsilon \to \hat{\mathbb{X}}^\varepsilon \in \mathcal{M}$, [BHZ16]
- Probabilistic step Convergence in probability of the renormalised model $\hat{\mathbb{X}}^\varepsilon$ to $\hat{\mathbb{X}}$ in $(\mathcal{M}, \mathrm{d})$, [CH16].

We obtain a renormalised solution $\hat{u} := \Phi(\hat{\mathbb{X}})$, also the unique solution of a fixed point problem.

This works for very general noises, far beyond the Gaussian case.

Wong-Zakai for SPDEs

The analogous result for the SPDE is much more subtle: if

$$\partial_t u_\varepsilon = \partial_x^2 u_\varepsilon + H(u_\varepsilon) + F(u_\varepsilon)\,\xi_\varepsilon, \qquad x \in \mathbb{R},$$

then $u_\varepsilon = \Phi(\mathbb{X}^\varepsilon)$ does not converge in general; necessary to renormalise the equation and study $\hat{u}_\varepsilon := \Phi(\hat{\mathbb{X}}^\varepsilon)$:

$$\partial_t \hat{u}_\varepsilon = \partial_x^2 \hat{u}_\varepsilon + \bar{H}(\hat{u}_\varepsilon) - C_\varepsilon\, F'(\hat{u}_\varepsilon)\, F(\hat{u}_\varepsilon) + F(\hat{u}_\varepsilon)\,\xi_\varepsilon$$

with $C_\varepsilon = \mathbb{E}[\xi_\varepsilon(G * \xi_\varepsilon)] \sim \varepsilon^{-1}$. The limit $\hat{u} := \Phi(\hat{\mathbb{X}})$ solves

$$\mathrm{d}\hat{u} = (\partial_x^2 \hat{u} + H(\hat{u}))\,\mathrm{d}t + F(\hat{u})\,\mathrm{d}W_t$$

in the Itô sense (true for very general ξ_ε, see [Chandra-Shen]).

Although there is nothing singular in this SPDE, the result is far from simple and requires the full power of the theory [Hairer-Pardoux15].

Important messages

We want to renormalise the (unknown) solution $u_\varepsilon = \Phi(\mathbb{X}^\varepsilon)$.

We renormalise the (finitely many, explicit) ill-defined products and construct the renormalised model $\hat{\mathbb{X}}^\varepsilon$ [BHZ16].

We prove that the renormalised model $\hat{\mathbb{X}}^\varepsilon$ converges to $\hat{\mathbb{X}}$ in $(\mathcal{M}, \mathrm{d})$ [CH16].

Continuity of the solution map $\Phi : \mathcal{M} \to \mathcal{S}'(\mathbb{R}^d)$ yields convergence of the renormalised solution $\hat{u}_\varepsilon = \Phi(\hat{\mathbb{X}}^\varepsilon)$ to $\hat{u} = \Phi(\hat{\mathbb{X}})$ [MH14].

Very important: $(\mathcal{M}, \mathrm{d})$, $\mathbb{X}^\varepsilon$, $\hat{\mathbb{X}}^\varepsilon$ and $\mathbb{X}^\varepsilon \to \hat{\mathbb{X}}^\varepsilon$ are all non-linear.

The group describing the transformation $\mathbb{X}^\varepsilon \to \hat{\mathbb{X}}^\varepsilon$ is in general non-commutative.

Renormalisation does not mean modifying the equation but choosing the correct equation.

Another example: KPZ

The regularised version is

$$\partial_t u_\varepsilon = \partial_x^2 u_\varepsilon + (\partial_x u_\varepsilon)^2 + \xi_\varepsilon$$

which has to be renormalised to

$$\partial_t \hat{u}_\varepsilon = \partial_x^2 \hat{u}_\varepsilon + (\partial_x \hat{u}_\varepsilon)^2 - C_\varepsilon + \xi_\varepsilon$$

and

$$C_\varepsilon = \mathbb{E}\left[(\partial_x G * \xi_\varepsilon)^2\right] \sim \frac{1}{\varepsilon}.$$

In this case, one of the ill-defined products to be renormalised is

$$(\partial_x G * \xi_\varepsilon)^2 \longrightarrow (\partial_x G * \xi_\varepsilon)^2 - \mathbb{E}[(\partial_x G * \xi_\varepsilon)^2].$$

Singular stochastic PDEs

Around 2010, Martin and Massimiliano, among others, try to generalise Rough Paths to stochastic PDEs like KPZ, PAM and Φ^4.

$$
\begin{aligned}
(\mathrm{KPZ}) \qquad & \partial_t u = \Delta u + (\nabla u)^2 + \xi, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \\
(\mathrm{PAM}) \qquad & \partial_t u = \Delta u + u\,\xi, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^2, \\
(\Phi^4_3) \qquad & \partial_t u = \Delta u - u^3 + \xi, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^3.
\end{aligned}
$$

This needs two generalisations:

- The rough path must be parametrized by $\mathbb{R}^d$ with $d \geq 2$
- $\mathbb{X}_{st}(\tau)$ can become a distribution, say, in t for fixed s, i.e. we want to allow that $\sup_{s\neq t}[|\mathbb{X}_{st}(\tau)|/|t-s|^{\alpha_\tau}] < +\infty$ with $\alpha_\tau \in \mathbb{R}$.

Two new theories are born: regularity structures and paraproducts.

Rough Paths ?

Consider e.g.

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + \sigma(u_\varepsilon)\,\xi_\varepsilon, \quad (t,x) \in \mathbb{R} \times \mathbb{R}.$$

What is the associated "Rough Path" (model) ? If we had before

$$\langle \mathbb{X}_{st}, [\tau]_i \rangle = \int_s^t (\mathbb{X}_{su}\tau)\,\dot{X}^i_u\,\mathrm{d}u$$

then now it looks reasonable to replace

$$\dot{X}^i_u \longrightarrow \xi_\varepsilon(u,y), \qquad \int_s^t \cdots\,\mathrm{d}u \longrightarrow \int_0^t \int_{\mathbb{R}} G_{t-u}(x-y)\,\cdots\,\mathrm{d}u\,\mathrm{d}y.$$

Rough Paths ?

In Rough Paths $\mathbb{X}_{st}(\tau)$ is always an increment

$$\begin{aligned}\langle \mathbb{X}_{st}, [\tau]_i \rangle &= \int_s^t (\mathbb{X}_{su}\tau)\, \dot{X}_u^i \,\mathrm{d}u \\ &= \int_a^t (\mathbb{X}_{su}\tau)\, \dot{X}_u^i \,\mathrm{d}u - \int_a^s (\mathbb{X}_{su}\tau)\, \dot{X}_u^i \,\mathrm{d}u.\end{aligned}$$

The analytic property

$$\sup_{s \neq t} [|\langle \mathbb{X}_{st}, \tau \rangle| / |t-s|^{\gamma|\tau|}] < +\infty$$

is recursive, since if s, t are close to each other then $u \in [s,t]$ is close to s as well.

Rough Paths ?

Let us use a now notation for the addition of a new trunk:

$$[\tau]_i \longrightarrow \mathcal{I}(\tau).$$

For SPDEs, we imagine a recursive object $\Pi_x \tau(y)$ replacing $\mathbb{X}_{st}(\tau)$, such that

$$\Pi_x \mathcal{I}(\tau)(y) = G * (\Pi_x \tau)(y) - G * (\Pi_x \tau)(x).$$

(From now on, x, y are space-time variables.)
What would be a reasonable analytic requirement here ? If

$$|\Pi_x \tau(y)| \leq C|y-x|^{|\tau|_\mathfrak{s}}$$

with $|\tau|_\mathfrak{s} > 0$ then we would like to have, by analogy with RPs,

$$|\Pi_x \mathcal{I}(\tau)(y)| \leq C|y-x|^{|\tau|_\mathfrak{s}+2}$$

but this requires further assumptions on $y \mapsto G * (\Pi_x \tau)(y)$.

Taylor sums and remainders

In fact we have to modify the definition of $\Pi_x\tau(y)$. We recall

$$\begin{aligned}\langle \mathbb{X}_{st}, [\tau]_i\rangle &= \int_s^t (\mathbb{X}_{su}\tau)\,\dot{X}^i_u\,\mathrm{d}u \\ &= \int_a^t (\mathbb{X}_{su}\tau)\,\dot{X}^i_u\,\mathrm{d}u - \int_a^s (\mathbb{X}_{su}\tau)\,\dot{X}^i_u\,\mathrm{d}u.\end{aligned}$$

This increment is a Taylor remainder at order 0. This suggests to go to a higher order by setting

$$\Pi_x\,\mathcal{I}(\tau)(y) = G * (\Pi_x\tau)(y) - \sum_{k\le|\mathcal{I}(\tau)|_{\mathfrak{s}}} \frac{(y-x)^k}{k!}\,\partial^k G * (\Pi_x\tau)(x).$$

But then we have to modify the coproduct if we want Chen's relation. It still involves extraction of a subtree at the root and contraction, but there are additional decorations that take into account the terms of the Taylor series.

Tree representation

Recall that we are interested in a finite number of polynomial functions of ξ_ε, $P_1(\xi_\varepsilon), \ldots, P_N(\xi_\varepsilon)$.

More precisely, for a fixed $\varphi \in C_c^\infty$ we consider the random variables

$$Z_i := \int_{\mathbb{R}^d} \varphi(z)\, P_i(\xi_\varepsilon(z))\, \mathrm{d}z, \qquad i = 1, \ldots, N.$$

To each such random variable we associate a rooted tree T_i.

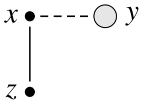
Every integration variable in Z_i is a vertex in T_i.

Every integral kernel in Z_i is an edge in T_i.

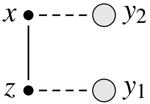
$$\Xi \longrightarrow \int \varphi(z)\, \xi_\varepsilon(z)\, \mathrm{d}z = \int \varphi(z)\, \rho_\varepsilon(z-x)\, \xi(\mathrm{d}x)\, \mathrm{d}z \quad \longrightarrow$$

Remark: the previous tree is absent in Rough Paths.

$$\mathcal{I}(\Xi) \longrightarrow \int \varphi(z)\, G * \xi_\varepsilon(z)\, \mathrm{d}z \quad \longrightarrow$$

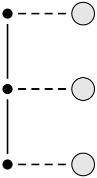


$$\Xi\mathcal{I}(\Xi) \longrightarrow \int \varphi(z)\, \xi_\varepsilon(z)\, G * \xi_\varepsilon(z)\, \mathrm{d}z \quad \longrightarrow$$

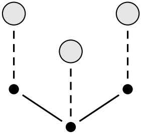


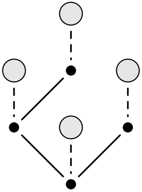
Examples

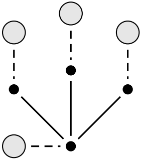
$\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi))$

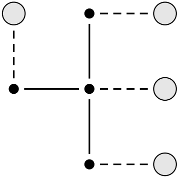


$\Xi\mathcal{I}(\Xi)^2$









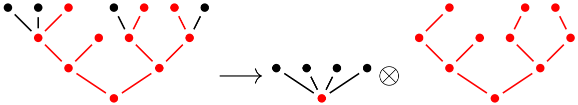
Further decorations on trees

We have additional decorations on trees, needed to code

$$\Pi_x \mathcal{I}(\tau)(y) = G * (\Pi_x \tau)(y) - \sum_{k \le |\mathcal{I}(\tau)|_{\mathfrak{s}}} \frac{(y-x)^k}{k!} \partial^k G * (\Pi_x \tau)(x).$$

- $\mathfrak{n}$ on nodes, representing powers of $(y-x)$
- $\mathfrak{e}$ on edges, representing derivatives $\partial^k G$ of the heat kernel

$$\Delta^+ T^{\mathfrak{n}}_{\mathfrak{e}} = \sum_{S \subseteq T} \sum_{\mathfrak{n}_S, \mathfrak{e}_S} \frac{1}{\mathfrak{e}_S!} \binom{\mathfrak{n}}{\mathfrak{n}_S} (T/S)^{\mathfrak{n}-\mathfrak{n}_S}_{\mathfrak{e}+\mathfrak{e}_S} \otimes S^{\mathfrak{n}_S+\pi\mathfrak{e}_S}_{\mathfrak{e}}$$



Distributions

We have a linear space $\mathcal{H}$ of decorated trees, representing distributions on $\mathbb{R}^d$ which are relevant to the given equation.

Since we do not expect to multiply all distributions, $\mathcal{H}$ is not assumed to be an algebra.

We do not expect $\mathcal{H}$ to have a coproduct either, so it is not clear how to define the Chen relation

$$\mathbb{X}_{xz} \star \mathbb{X}_{zy} = \mathbb{X}_{xy}.$$

The solution is to split $\mathbb{X}_{xy}$ into two components, containing respectively functions and distributions.

Remember: in Rough Paths we have $\mathbb{X}_{st} = \mathbb{X}_s^{-1} \star \mathbb{X}_t$.

Then we want to differentiate the two factors, and have $\mathbb{X}_s^{-1}$ behaving as a true function of s, while $\mathbb{X}_t$ can behave as a distribution in t.

Comodules

We consider two spaces of decorated trees, $\mathcal{H}$ and $\mathcal{H}_+$ such that

- $\mathcal{H}_+$ is a Hopf algebra and codes classical functions
- $\mathcal{H}$ is a linear space coding relevant explicit distributions
- we have a left coaction

$$\Delta^+ : \mathcal{H} \to \mathcal{H}_+ \otimes \mathcal{H}$$

compatible with the coproduct of $\mathcal{H}_+$.

Then $\mathcal{H}$ is a comodule over $\mathcal{H}_+$.

For $g_x \in \mathcal{G}_+$ and $\Pi : \mathcal{H} \to \mathcal{S}'(\mathbb{R}^d)$,

$$\Pi_x \tau(y) := \langle g_x \otimes \Pi, \Delta^+ \tau \rangle(y)$$

is a good candidate for $\mathbb{X}_{xy} = \mathbb{X}_x^{-1} \star \mathbb{X}_y$.

Remarks

$$\Pi_x \tau(y) = \langle g_x \otimes \Pi, \Delta^+ \tau \rangle (y)$$

- $g_x \in \mathcal{G}_+$ is a character and therefore multiplicative
- in general $\Pi : \mathcal{H} \to \mathcal{S}'(\mathbb{R}^d)$ is not multiplicative, even if it takes values in smooth functions
- this "freedom" of Π to be non-multiplicative is crucial in the renormalisation procedure
- Π is always assumed to satisfy

$$\Pi \Xi = \xi_\varepsilon, \qquad \Pi \mathcal{I}(\tau) = G * \Pi \tau$$

- the canonical choice of Π, for a regularised version ξ_ε of the noise, satisfies moreover multiplicativity

$$\Pi(\tau_1 \cdots \tau_n) = \Pi(\tau_1) \cdots \Pi(\tau_n).$$

Renormalisation

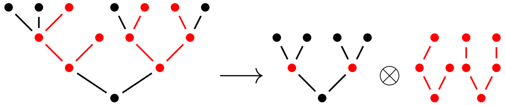
We consider a third space of decorated forests, $\mathcal{H}_-$

- $\mathcal{H}_-$ is a Hopf algebra and codes renormalisation of diverging subtrees
- we have right coactions

$$\Delta^- : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}_-, \qquad \Delta^- : \mathcal{H}_+ \to \mathcal{H}_+ \otimes \mathcal{H}_-$$

compatible with the coproduct of $\mathcal{H}_-$, so that $\mathcal{H}$ and $\mathcal{H}_+$ are comodules over $\mathcal{H}_-$.

$$\Delta^- T^{\mathfrak{n}}_{\mathfrak{e}} = \sum_S \sum_{\mathfrak{n}_S, \mathfrak{e}_S} \frac{1}{\mathfrak{e}_S!} \binom{\mathfrak{n}}{\mathfrak{n}_S} (T/S)^{\mathfrak{n}-\mathfrak{n}_S}_{\mathfrak{e}+\mathfrak{e}_S} \otimes S^{\mathfrak{n}_S+\pi \mathfrak{e}_S}_{\mathfrak{e}}$$



Positive and negative renormalisations

If we set

- $\mathfrak{A}^+(T) := \{S \subseteq T : S \text{ subtree with the same root as } T\}$
- $\mathfrak{A}^-(T) := \{S \subseteq T : S \text{ subforest of } T\}$

then

$$\Delta^+ T^{\mathfrak{n}}_{\mathfrak{e}} = \sum_{S \in \mathfrak{A}^+(T)} \sum_{\mathfrak{n}_S, \mathfrak{e}_S} \frac{1}{\mathfrak{e}_S!} \binom{\mathfrak{n}}{\mathfrak{n}_S} (T/S)^{\mathfrak{n}-\mathfrak{n}_S}_{\mathfrak{e}+\mathfrak{e}_S} \otimes S^{\mathfrak{n}_S+\pi\mathfrak{e}_S}_{\mathfrak{e}}$$

$$\Delta^- T^{\mathfrak{n}}_{\mathfrak{e}} = \sum_{S \in \mathfrak{A}^-(T)} \sum_{\mathfrak{n}_S, \mathfrak{e}_S} \frac{1}{\mathfrak{e}_S!} \binom{\mathfrak{n}}{\mathfrak{n}_S} (T/S)^{\mathfrak{n}-\mathfrak{n}_S}_{\mathfrak{e}+\mathfrak{e}_S} \otimes S^{\mathfrak{n}_S+\pi\mathfrak{e}_S}_{\mathfrak{e}}$$

The renormalised model

We define for $\ell \in \mathcal{G}_- \subset \mathcal{H}_-^*$ maps $M_\ell : \mathcal{H} \to \mathcal{H}$ and $M_\ell : \mathcal{H}_+ \to \mathcal{H}_+$

$$M_\ell(\tau) := (\mathrm{id} \otimes \ell)\Delta^- \tau.$$

We can define for $\ell \in \mathcal{G}_- \subset \mathcal{H}_-^*$

$$\begin{aligned} \Pi_x^\ell \tau(y) &:= \langle g_x M_\ell \otimes \Pi M_\ell, \Delta^+ \tau \rangle (y) \\ &= (g_x \otimes \ell \otimes \Pi \otimes \ell)(\Delta^- \otimes \Delta^-)\Delta^+ \tau(y). \end{aligned}$$

A compatibility condition between these coactions implies that this works well...

$\mathcal{G}_+$ is the structure group, $\mathcal{G}_-$ the renormalisation group.

Coactions and coproducts

We have defined 2 coproducts and 3 coactions, which are all variants of just 2 operators Δ^+, Δ^-:

- a contraction/extraction of subtrees at the root (as in Rough Paths)
- a contraction/extraction of subforests.

We also have a non-trivial action on decorations, related to the Taylor sums, which is the same for all operators.

For the Analytical theory: there is an analog of controlled paths.

Several theorems replace the Sewing Lemma.