COURSE "FLUCTUATIONS OF CHAOTIC RANDOM VARIABLES: THEORETICAL FOUNDATIONS AND GEOMETRIC APPLICATIONS"

GIOVANNI PECCATI

In recent years, a large body of work has been devoted to the study of the asymptotic properties of those random variables that belong to the homogeneous chaos of some random measure, typically by means of variational techniques and integration by parts formulae. The aim of these lectures is to present some of the crucial aspects of the theory, by specifically focussing on combinatorial aspects, that are in particular tightly connected to the Rota–Wallstrom theory of combinatorial integration. Such a theory of integration, appearing in the fundamental paper [4] provides a unified combinatorial framework, in order to understand (multiple) stochastic integration and associated formulae as consequences of the properties of partition lattices and associated Möbius inversion formulae. Some of the topics that we will deal with in the lectures involve fourth moment theorems, asymptotic independence, Edgeworth expansions, as well as a the introduction to a recent stream of research, focussing on probabilistic approximations by means of Markov generators and carré-du-champ operators. If time permits, we will also illustrate three remarkable geometric applications: to random geometric graphs, to nodal sets of random waves, and to real polarization problems. The last topic is intimately connected to a long-standing open problem in Gaussian analysis, known as the "Gaussian product conjecture", that we will try to illustrate once again from a combinatorial standpoint.

References:

- [1] I. Nourdin, G. Peccati (2012),
  Normal Approximations Using Malliavin Calculus: from Stein's Method to Universality, Cambridge Tracts in Mathematics. Cambridge University Press.
- [2] G. Peccati, M. Reitzner (Editors) (2016), Stochastic analysis for Poisson point processes: Malliavin calculus, Wiener-Itô chaos expansions and stochastic geometry, Springer-Verlag.
- [3] G. Peccati, M.S. Taqqu (2010), Wiener Chaos: Moments, Cumulants and Diagrams, Springer-Verlag.
- [4] G.-C. Rota, T. C. Wallstrom (1997), *Stochastic integrals: a combinatorial approach*, Ann. Probab. 25, (3): 1257-1283

COURSE "ALGEBRAIC AND COMBINATORIAL ASPECTS IN STOCHASTIC CALCULUS"

KURUSCH EBRAHIMI-FARD & FRÉDÉRIC PATRAS

In the last decade Hopf and Lie algebraic structures have reshaped the field of algebraic combinatorics. Moreover, they turn out to be crucial in the context of stochastic calculus. These developments can be traced back to G.-C. Rota's work, who uncovered beautiful links between the combinatorics of the poset of set partitions, the related Möbius calculus as well as combinatorial bialgebras, and various probabilistic and stochastic structures and phenomena, appearing, for instance, in cumulant calculus and the fine properties of stochastic integrals.

In this course we will discuss in detail shuffle and quasi-shuffle Hopf algebras together with related (pre-)Lie algebraic aspects, in the context of computations involving iterated stochastic integrals in the Stratonovich and Itô calculus framework. A particular emphasis will be put on the interplay between Hopf algebra structures (and generalizations thereof) on permutations, surjections, set partitions and rooted trees. We will extend the scope by exploring similar combinatorial and algebraic structures, which have been unveiled in the context of moment-cumulant calculus in noncommutative probability theory.

References:

- L. Foissy, F. Patras, J.-Y. Thibon (2016),
  Deformations of shuffles and quasi-shuffles, Annales Inst. Fourier, 66, no 1 (2016).
- K. Ebrahimi-Fard, F. Patras (2016),
  The splitting process in free probability, IMRN (2016) 2016 (9): 2647-2676. arXiv:1502.02748v1.
- K. Ebrahimi-Fard, S.J.A. Malham, F. Patras, A. Wiese (2015),
  The exponential Lie series for continuous semimartingales, Proc. R. Soc. A: 2015, 471, Issue: 2184. arXiv:1506.07686
- K. Ebrahimi-Fard, S.J.A. Malham, F. Patras, A. Wiese (2015),
  Flows and stochastic Taylor series in Ito calculus, J. Phys. A: Math. Theor. 48 495202 (2015). arXiv:1504.07226
- K. Ebrahimi-Fard, F. Patras (2015),
  Cumulants, free cumulants and half-shuffles Proc. R. Soc. A: 2015, 471, Issue: 2176. arXiv:1409.5664
- K. Ebrahimi-Fard, F. Patras (arXiv:1701.06152),
  Monotone, free, and boolean cumulants: a shuffle algebra approach.
- K. Ebrahimi-Fard, F. Patras (2013),
  La structure combinatoire du calcul intégral, Gazette des Mathématiciens 138, Oct. 2013, (SMF).
- M. Hazewinkel, N. Gubareni, V. Kirichenko (2010),
  Algebras, Rings and Modules: Lie Algebras and Hopf Algebras, Mathematical Surveys and Monographs, Vol. 168, AMS
- Ph. E. Protter (2005)
  Stochastic Integration and Differential Equations, 2nd Ed., Springer-Verlag Berlin Heidelberg.

COURSE "COMBINATORIAL ASPECTS OF FREE PROBABILITY AND FREE STOCHASTIC CALCULUS"

ROLAND SPEICHER

Free probability was initiated by Voiculescu in the 1980s in the context of operator algebras (in particular, the question about isomorphism of von Neumann algebras), and received a tremendous boost, when Voiculescu discovered in 1991 its relations to random matrices. Since then it has evolved into a very active subject with many relations to other fields. One particular feature of free probability is its combinatorial structure, which goes in parallel to the combinatorics in classical probability theory - one only has to replace the lattice of all partition by the lattice of non-crossing partitions.

The first lectures will introduce the basics of free probability from this combinatorial perspective. The lattice of non-crossing partitions will be used to define the notion of free cumulants. This will then allow us to deal with the additive and multiplicative free convolution.

Via a combinatorial treatment of the free central limit theorem we will then recognize the semicircular distribution as the free analogue of the Gaussian distribution. This will be pushed further to the definition of the free analogue of the Brownian motion. We will see free analogues of Ito's formulas and also of Malliavin calculus; again an emphasis will be on a combinatorial understanding of the difference between the classical and the free situation.

Finally, we will also address some recent rough path approaches to free, and more generally non-commutative, stochastic integrations.

References:

- Roland Speicher (2017),
  Free probability theory and its atavars in representation theory, random matrices, and operator algebras; also featuring: non-commutative distributions, Jahresber. Dtsch. Math. Ver 119 (2017), 3-30.
- Alexandru Nica, Roland Speicher (2006),
  Lectures on the Combinatorics of Free Probability, London Mathematical Society Lecture Notes, vol. 335, Cambridge University Press, 2006.
- Philippe Biane, Roland Speicher (1998),
  Stochastic calculus with respect to free Brownian motion and analysis on Wigner space, Prob. Th. Rel. Fields 112 (1998), 373-410
- Todd Kemp, Ivan Nourdin, Giovanni Peccati, Roland Speicher (2012),
  Wigner chaos and the fourth moment, Ann. Probab. 40 (2012), 1577-1635.
- Aurélien Deya, René Schott (2013)
  On the rough-paths approach to non-commutative stochastic calculus, J. Funct. Anal. 265 (2013): 594-628.

COURSE "ROUGH PATHS, REGULARITY STRUCTURES AND RENORMALISATION"

LORENZO ZAMBOTTI

I will start from Kuo-Tsai Chen's theory of iterated integrals of a smooth path in $\mathbb{R}^d$, showing how they define naturally a family of characters on a tensor algebra, satisfying a flow property. This beautiful result is the basis of Terry Lyons' idea to define a geometric rough path, which retains the algebraic property found by Chen but not the smoothness of the underlying path, which is instead replaced by a mild analytic condition. The motivation of this theory is to provide a pathwise approach to stochastic integration and stochastic differential equations.

The next topic is branched rough paths, which have been introduced by Massimiliano Gubinelli as an extension of geometric rough paths. Now the relevant algebraic structure is not a tensor algebra but a space of rooted trees/forests, the Connes-Kreimer Hopf algebra. I will briefly explain the definition of controlled rough paths, again due to Gubinelli.

Finally we'll discuss regularity structures, a far-reaching extension of branched rough paths, whose motivation is solving stochastic partial differential equations. The main novelties with respect to rough paths are at least three: the equations involve functions with a parameter in $\mathbb{R}^d$ rather than $\mathbb{R}$; non-linear functions of distributions (generalised functions) appear in the equations; infinite quantities have to be renormalised.

In this situation the algebraic setting becomes more complex and we need to consider two Hopf algebras rather than one, whose algebraic structures are compatible in the sense that they are in cointeraction, as in the work on B-series of Calaque/Ebrahimi-Fard/Manchon. Furthermore, we have a third linear space with a structure of co-module over the two Hopf algebras. Our spaces are coded by trees/forests but with additional decorations which are needed in order to describe appropriate Taylor expansions. The main focus is rather on the associated character groups: one is called the Structure group, the other the Renormalisation group.

References:

- Massimiliano Gubinelli (2004),
  Ramification of rough paths, J. Differential Equations
- Massimiliano Gubinelli (2010),
  Controlling rough paths, J. Funct. Anal.
- David Kelly, Martin Hairer (2015),
  Geometric versus non-geometric rough paths, Ann. Inst. Henri Poincar Probab. Stat.
- Martin Hairer (2014),
  A theory of regularity structures, Inventiones.
- Yvain Bruned, Martin Hairer, L.Z. (2016),
  Algebraic renormalisation of regularity structures, arXiv.
- Ajay Chandra, Martin Hairer (2016),
  An analytic BPHZ theorem for regulariy structures, arXiv.